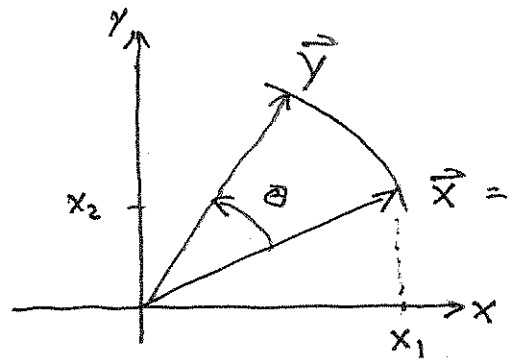
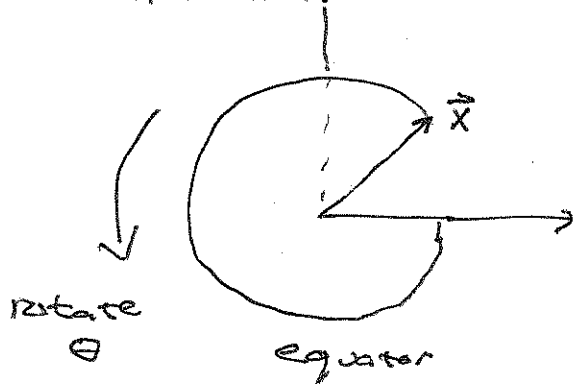
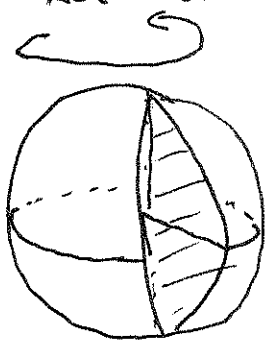


# 2.1: Linear Transformations

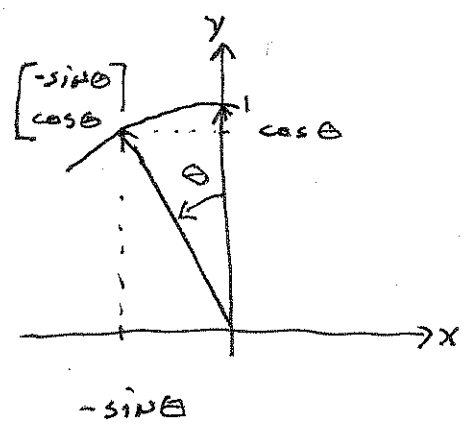
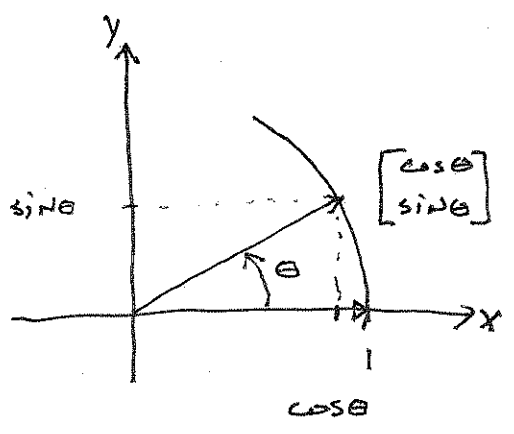
Ex1: Rotation using Mathematica



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\vec{e}_1$                        $\vec{e}_2$

Goal: Find  $T(\vec{x})$  ... the fun that rotates  $\vec{x}$  by  $\theta$  C.C.W.  
 Let's rotate  $\vec{e}_1$  &  $\vec{e}_2$  C.C.W. by the angle  $\theta$ .



$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \rightarrow \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \rightarrow \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

$$\begin{aligned}
T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\
&= x_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + x_2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= A\vec{x}
\end{aligned}$$

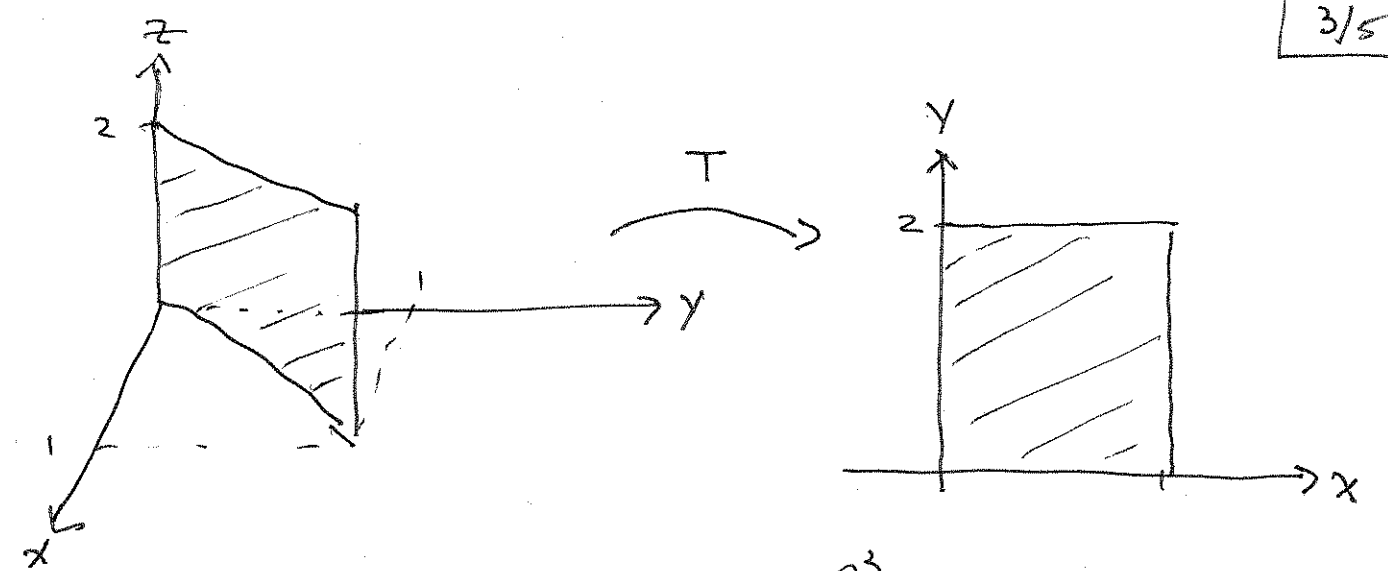
This rotation fct  $T$  is an example of a linear transformation.

DEF: A fct  $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is called a linear transformation if  $\exists A_{n \times m}$  s.t.  $T(\vec{x}) = A\vec{x}$   
 $\forall \vec{x} \in \mathbb{R}^m$ .

Ex 2: Consider the vectors  $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

under the transformation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



projects an image in  $\mathbb{R}^3$  onto the plane.

Some transforms can be undone (ex1) and others can't (ex2). That is, the inverse of a linear transformation doesn't always exist.

The identity matrix/transformation.

The standard vectors  $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$

Thm: If  $T$  is a linear transformation, then the corresponding transformation matrix is.

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \dots & T(\vec{e}_m) \\ | & & | \end{bmatrix}_{n \times m}$$

To see this: If  $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix}$ , then  $A\vec{e}_i = \vec{v}_i$   
for  $i=1, \dots, m$

recall  $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$   
 and  $A(k\vec{v}) = kA\vec{v}$ .

Thm: A transformation  $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is linear  
 iff

- (a)  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$
- (b)  $T(k\vec{v}) = kT(\vec{v})$ .

□ proof.

( $\Rightarrow$ ) For (b) (assume  $T$  is linear).

$$T(k\vec{v}) = A(k\vec{v}) = kA\vec{v} = kT(\vec{v}).$$

(a) is nearly identical.

( $\Leftarrow$ ) (assume properties (a) & (b)).

consider  $T(\vec{x})$  where  $\vec{x} \in \mathbb{R}^m$ .

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) \\ &= T(x_1\vec{e}_1) + \dots + T(x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + \dots + x_nT(\vec{e}_n) \\ &= \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \dots & T(\vec{e}_n) \\ | & & | \end{bmatrix} \vec{x} \\ &= A\vec{x}. \quad \blacksquare \end{aligned}$$

2.2  
5/5

What happens to  $\vec{x}$  under  
the linear transformation  $T$   
if  $T(\vec{v}_1) = 2\vec{v}_1$  &  $T(\vec{v}_2) = \frac{4}{3}\vec{v}_2$ ?

