2.4.108 a The formula $\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1-Rk & L+R-kLR \\ -k & 1-kL \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}$ is given, which implies that

$y = (1 - Rk)x + (L + R - kLR)m.$

In order for y to be independent of x it is required that $1 - Rk = 0$, or $k = \frac{1}{R} = 40$ (diopters).

$\frac{1}{k}$ then equals R, which is the distance between the plane of the lens and the plane on which parallel incoming rays focus at a point; thus the term "focal length" for $\frac{1}{k}$.

b Now we want y to be independent of the slope m (it must depend on x alone). In view of the formula above, this is the case if $L + R - kLR = 0$, or $k = \dfrac{L+R}{LR} = \dfrac{1}{R} + \dfrac{1}{L} = 40 + \dfrac{10}{3} \approx 43.3$ (diopters).

c Here the transformation is

$$\begin{bmatrix} y \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & D \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1-k_1D & D \\ k_1k_2D - k_1 - k_2 & 1-k_2D \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

We want the slope n of the outgoing rays to depend on the slope m of the incoming rays alone, and not on x; this forces $k_1k_2D - k_1 - k_2 = 0$, or, $D = \frac{k_1+k_2}{k_1k_2} = \frac{1}{k_1} + \frac{1}{k_2}$, the sum of the focal lengths of the two lenses. See Figure 2.68.

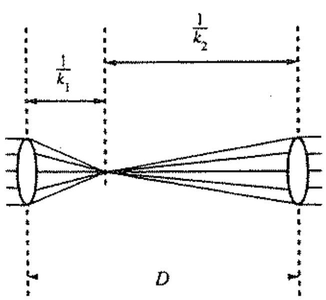

Figure 2.68: for Problem 2.4.108c.

True or False

Ch 2.TF.**1** T, by Theorem 2.4.3.

Ch 2.TF.**2** T; Let $A = B$ in Theorem 2.4.7.

Ch 2.TF.**3** F, by Theorem 2.3.3.

Ch 2.TF.**4** T, by Theorem 2.4.8.

Ch 2.TF.**5** F; Matrix AB will be 3×5, by Definition 2.3.1b.

Ch 2.TF.**6** F; Note that $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A linear transformation transforms $\vec{0}$ into $\vec{0}$.

Ch 2.TF.**7** T, by Theorem 2.2.4.

Ch 2.TF.**8** T, by Theorem 2.4.6.

Ch 2.TF.**9** T; The matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Ch 2.TF.**10** F; The columns of a rotation matrix are unit vectors; see Theorem 2.2.3.

Ch 2.TF.**11** F; Note that $\det(A) = (k-2)^2 + 9$ is always positive, so that A is invertible for all values of k.

Ch 2.TF.**12** T; Note that the columns are unit vectors, since $(-0.6)^2 + (\pm 0.8)^2 = 1$. The matrix has the form presented in Theorem 2.2.3.

Ch 2.TF.**13** F; Consider $A = I_2$ (or any other invertible 2×2 matrix).

Ch 2.TF.**14** T; Note that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}^{-1}$ is the unique solution.

Ch 2.TF.**15** F, by Theorem 2.4.9. Note that the determinant is 0.

Ch 2.TF.**16** T, by Theorem 2.4.3.

Ch 2.TF.**17** T; The shear matrix $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ works.

Ch 2.TF.**18** T; Simplify to see that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4y \\ -12x \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Ch 2.TF.**19** T; The equation $\det(A) = k^2 - 6k + 10 = 0$ has no real solution.

Ch 2.TF.**20** T; The matrix fails to be invertible for $k = 5$ and $k = -1$, since the determinant $\det A = k^2 - 4k - 5 = (k-5)(k+1)$ is 0 for these values of k.

Ch 2.TF.**21** T; The product is $\det(A)I_2$.

Ch 2.TF.**22** T; Writing an upper triangular matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and solving the equation $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ we find that $A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, where b is any nonzero constant.

Ch 2.TF.**23** T; Note that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a rotation through $\pi/2$. Thus $n = 4$ (or any multiple of 4) works.

Ch 2.TF.**24** F; If a matrix A is invertible, then so is A^{-1}. But $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ fails to be invertible.

Ch 2.TF.**25** F; If matrix A has two identical rows, then so does AB, for any matrix B. Thus AB cannot be I_n, so that A fails to be invertible.

Ch 2.TF.**26** T, by Theorem 2.4.8. Note that $A^{-1} = A$ in this case.

Ch 2.TF.**27** F; For any 2×2 matrix A, the two columns of $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ will be identical.

Ch 2.TF.**28** T; One solution is $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Ch 2.TF.**29** F; A reflection matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Here, $a^2 + b^2 = 1 + 1 = 2$.

Ch 2.TF.**30** T; Just multiply it out.

Ch 2.TF.**31** F; Consider matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, for example.

Ch 2.TF.**32** T; Apply Theorem 2.4.8 to the equation $(A^2)^{-1}AA = I_n$, with $B = (A^2)^{-1}A$.

Ch 2.TF.**33** F; Consider the matrix A that represents a rotation through the angle $2\pi/17$.

Ch 2.TF.**34** F; Consider the reflection matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Ch 2.TF.**35** T; We have $(5A)^{-1} = \frac{1}{5}A^{-1}$.

Ch 2.TF.**36** T; The equation $A\vec{e}_i = B\vec{e}_i$ means that the ith columns of A and B are identical. This observation applies to all the columns.

Ch 2.TF.**37** T; Note that $A^2B = AAB = ABA = BAA = BA^2$.

Ch 2.TF.**38** T; Multiply both sides of the equation $A^2 = A$ with A^{-1}.

Ch 2.TF.**39** F; Consider $A = I_2$ and $B = -I_2$.

Ch 2.TF.**40** T; Since $A\vec{x}$ is on the line onto which we project, the vector $A\vec{x}$ remains unchanged when we project again: $A(A\vec{x}) = A\vec{x}$, or $A^2\vec{x} = A\vec{x}$, for all $\vec{x}$. Thus $A^2 = A$.

Ch 2.TF.**41** T; If you reflect twice in a row (about the same line), you will get the original vector back: $A(A\vec{x}) = \vec{x}$, or, $A^2\vec{x} = \vec{x} = I_2\vec{x}$. Thus $A^2 = I_2$ and $A^{-1} = A$.

Ch 2.TF.**42** F; Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, for example.

Ch 2.TF.43 T; Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, for example.

Ch 2.TF.44 F; By Theorem 1.3.3, there is a nonzero vector $\vec{x}$ such that $B\vec{x} = \vec{0}$, so that $AB\vec{x} = \vec{0}$ as well. But $I_3\vec{x} = \vec{x} \neq \vec{0}$, so that $AB \neq I_3$.

Ch 2.TF.45 T; We can rewrite the given equation as $A^2 + 3A = -4I_3$ and $-\frac{1}{4}(A + 3I_3)A = I_3$. By Theorem 2.4.8, the matrix A is invertible, with $A^{-1} = -\frac{1}{4}(A + 3I_3)$.

Ch 2.TF.46 T; Note that $(I_n + A)(I_n - A) = I_n^2 - A^2 = I_n$, so that $(I_n + A)^{-1} = I_n - A$.

Ch 2.TF.47 F; A and C can be two matrices which fail to commute, and B could be I_n, which commutes with anything.

Ch 2.TF.48 F; Consider $T(\vec{x}) = 2\vec{x}$, $\vec{v} = \vec{e}_1$, and $\vec{w} = \vec{e}_2$.

Ch 2.TF.49 F; Since there are only eight entries that are not 1, there will be at least two rows that contain only ones. Having two identical rows, the matrix fails to be invertible.

Ch 2.TF.50 F; Let $A = B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, for example.

Ch 2.TF.51 F; We will show that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S$ fails to be diagonal, for an arbitrary invertible matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$. Since c and d cannot both be zero (as S must be invertible), at least one of the off-diagonal entries ($-c^2$ and d^2) is nonzero, proving the claim.

Ch 2.TF.52 T; Consider an $\vec{x}$ such that $A^2\vec{x} = \vec{b}$, and let $\vec{x}_0 = A\vec{x}$. Then $A\vec{x}_0 = A(A\vec{x}) = A^2\vec{x} = \vec{b}$, as required.

Ch 2.TF.53 T; Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now we want $A^{-1} = -A$, or $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. This holds if $ad - bc = 1$ and $d = -a$. These equations have many solutions: for example, $a = d = 0, b = 1, c = -1$. More generally, we can choose an arbitrary a and an arbitrary nonzero b. Then, $d = -a$ and $c = -\frac{1+a^2}{b}$.

Ch 2.TF.54 F; Consider a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We make an attempt to solve the equation $A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a + d) \\ c(a + d) & d^2 + bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Now the equation $b(a + d) = 0$ implies that $b = 0$ or $d = -a$.

If $b = 0$, then the equation $d^2 + bc = -1$ cannot be solved.

If $d = -a$, then the two diagonal entries of A^2, $a^2 + bc$ and $d^2 + bc$, will be equal, so that the equations $a^2 + bc = 1$ and $d^2 + bc = -1$ cannot be solved simultaneously.

In summary, the equation $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ cannot be solved.

Ch 2.TF.55 T; Recall from Definition 2.2.1 that a projection matrix has the form $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$, where $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a unit vector. Thus, $a^2 + b^2 + c^2 + d^2 = u_1^4 + (u_1u_2)^2 + (u_1u_2)^2 + u_2^4 = u_1^4 + 2(u_1u_2)^2 + u_2^4 = (u_1^2 + u_2^2)^2 = 1^2 = 1$.

Ch 2.TF.56 T; We observe that the systems $AB\vec{x} = 0$ and $B\vec{x} = 0$ have the same solutions (multiply with A^{-1} and A, respectively, to obtain one system from the other). Then, by True or False Exercise 45 in Chapter 1, rref(AB) =rref(B).