this process until you run out of variables or equations. Consider the example discussed on page 2:

$$\begin{vmatrix} x + 2y + 3z = 39 \\ x + 3y + 2z = 34 \\ 3x + 2y + z = 26 \end{vmatrix}.$$

We can solve the first equation for x:

$$x = 39 - 2y - 3z.$$

Then we substitute this equation into the other equations:

$$\begin{vmatrix} (39 - 2y - 3z) + 3y + 2z = 34 \\ 3(39 - 2y - 3z) + 2y + z = 26 \end{vmatrix}.$$

We can simplify:

$$\begin{vmatrix} y - z = -5 \\ -4y - 8z = -91 \end{vmatrix}.$$

Now, $y = z - 5$, so that $-4(z - 5) - 8z = -91$, or

$$-12z = -111.$$

We find that $z = \frac{111}{12} = 9.25$. Then

$$y = z - 5 = 4.25,$$

and

$$x = 39 - 2y - 3z = 2.75.$$

Explain why this method is essentially the same as the method discussed in this section; only the bookkeeping is different.

46. A hermit eats only two kinds of food: brown rice and yogurt. The rice contains 3 grams of protein and 30 grams of carbohydrates per serving, while the yogurt contains 12 grams of protein and 20 grams of carbohydrates.

a. If the hermit wants to take in 60 grams of protein and 300 grams of carbohydrates per day, how many servings of each item should he consume?

b. If the hermit wants to take in P grams of protein and C grams of carbohydrates per day, how many servings of each item should he consume?

47. I have 32 bills in my wallet, in the denominations of US$ 1, 5, and 10, worth $100 in total. How many do I have of each denomination?

48. Some parking meters in Milan, Italy, accept coins in the denominations of 20¢, 50¢, and € 2. As an incentive program, the city administrators offer a big reward (a brand new Ferrari Testarossa) to any meter maid who brings back exactly 1,000 coins worth exactly € 1,000 from the daily rounds. What are the odds of this reward being claimed anytime soon?

1.2 Matrices, Vectors, and Gauss–Jordan Elimination

When mathematicians in ancient China had to solve a system of simultaneous linear equations such as[4]

$$\begin{vmatrix} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{vmatrix},$$

they took all the numbers involved in this system and arranged them in a rectangular pattern (*Fang Cheng* in Chinese), as follows:[5]

3	21	−3	0
−6	−2	−1	62
2	−3	8	32

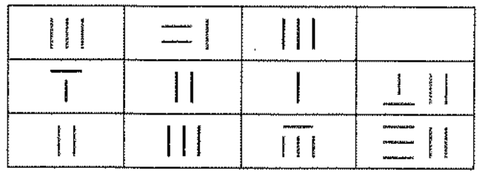
All the information about this system is conveniently stored in this array of numbers.

The entries were represented by bamboo rods, as shown below; red and black rods stand for positive and negative numbers, respectively. (Can you detect how this

[4]This example is taken from Chapter 8 of the *Nine Chapters on the Mathematical Art*; see page 1. Our source is George Gheverghese Joseph, *The Crest of the Peacock, Non-European Roots of Mathematics*, 2nd ed., Princeton University Press, 2000.

[5]Actually, the roles of rows and columns were reversed in the Chinese representation.

number system works?) The equations were then solved in a hands-on fashion, by manipulating the rods. We leave it to the reader to find the solution.



Today, such a rectangular array of numbers,

$$\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix},$$

is called a *matrix*.[6] Since this particular matrix has three rows and four columns, it is called a 3×4 matrix ("three by four").

The four columns of the matrix

The three rows of the matrix $\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix}$

Note that the first column of this matrix corresponds to the first variable of the system, while the first row corresponds to the first equation.

It is customary to label the entries of a 3×4 matrix A with double subscripts as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

The first subscript refers to the row, and the second to the column: The entry a_{ij} is located in the ith row and the jth column.

Two matrices A and B are equal if they are the same size and if corresponding entries are equal: $a_{ij} = b_{ij}$.

If the number of rows of a matrix A equals the number of columns (A is $n \times n$), then A is called a *square matrix*, and the entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the (main) *diagonal* of A. A square matrix A is called *diagonal* if all its entries above and below the main diagonal are zero; that is, $a_{ij} = 0$ whenever $i \neq j$. A square matrix A is called *upper triangular* if all its entries below the main diagonal are zero; that is, $a_{ij} = 0$ whenever i exceeds j. *Lower triangular* matrices are defined analogously. A matrix whose entries are all zero is called a *zero matrix* and is denoted by 0 (regardless of its size). Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 0 & 0 \\ 4 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

[6]It appears that the term *matrix* was first used in this sense by the English mathematician J. J. Sylvester, in 1850.

The matrices B, C, D, and E are square, C is diagonal, C and D are upper triangular, and C and E are lower triangular.

Matrices with only one column or row are of particular interest.

Vectors and vector spaces

A matrix with only one column is called a column vector, or simply a vector. The entries of a vector are called its components. The set of all column vectors with n components is denoted by $\mathbb{R}^n$; we will refer to $\mathbb{R}^n$ as a *vector space*.

A matrix with only one row is called a row vector.

In this text, the term *vector* refers to column vectors, unless otherwise stated. The reason for our preference for column vectors will become apparent in the next section.

Examples of vectors are

$$\begin{bmatrix} 1 \\ 2 \\ 9 \\ 1 \end{bmatrix},$$

a (column) vector in $\mathbb{R}^4$, and

$$\begin{bmatrix} 1 & 5 & 5 & 3 & 7 \end{bmatrix},$$

a row vector with five components. Note that the m columns of an $n \times m$ matrix are vectors in $\mathbb{R}^n$.

In previous courses in mathematics or physics, you may have thought about vectors from a more geometric point of view. (See the Appendix for a summary of basic facts on vectors.) Let's establish some conventions regarding the geometric representation of vectors.

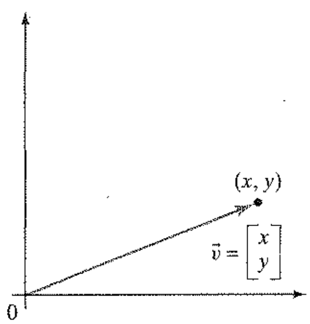


Figure 1

Standard representation of vectors

The standard representation of a vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

in the Cartesian coordinate plane is as an *arrow* (a directed line segment) from the origin to the point (x, y), as shown in Figure 1.

The standard representation of a vector in $\mathbb{R}^3$ is defined analogously.

In this text, we will consider the standard representation of vectors, unless stated otherwise.

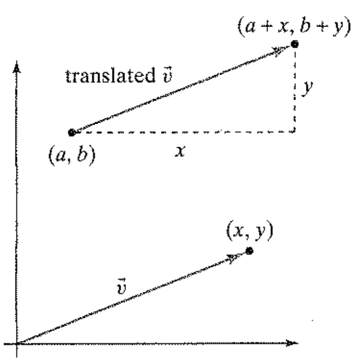


Figure 2

Occasionally, it is helpful to translate (or shift) the vector in the plane (preserving its direction and length), so that it will connect some point (a, b) to the point $(a + x, b + y)$, as shown in Figure 2.

When considering an infinite set of vectors, the arrow representation becomes impractical. In this case, it is sensible to represent the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ simply by the point (x, y), the head of the standard arrow representation of $\vec{v}$.

For example, the set of all vectors $\vec{v} = \begin{bmatrix} x \\ x + 1 \end{bmatrix}$ (where x is arbitrary) can be represented as the *line* $y = x + 1$. For a few special values of x we may still use the arrow representation, as illustrated in Figure 3.

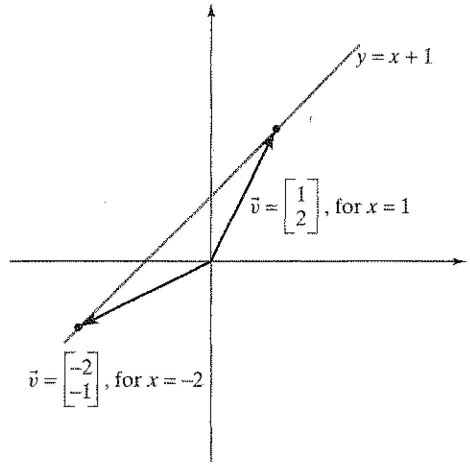


Figure 3

In this course, it will often be helpful to think about a vector numerically, as a list of numbers, which we will usually write in a column.

In our digital age, information is often transmitted and stored as a string of numbers (i.e., as a vector). A section of 10 seconds of music on a CD is stored as a vector with 440,000 components. A weather photograph taken by a satellite is transmitted to Earth as a string of numbers.

Consider the system

$$\begin{vmatrix} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{vmatrix}.$$

Sometimes we are interested in the matrix

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix},$$

which contains the coefficients of the system, called its *coefficient matrix*.

By contrast, the matrix

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix},$$

which displays all the numerical information contained in the system, is called its *augmented matrix*. For the sake of clarity, we will often indicate the position of the equal signs in the equations by a dotted line:

$$\left[\begin{array}{ccc:c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right].$$

To solve the system, it is more efficient to perform the elimination on the augmented matrix rather than on the equations themselves. Conceptually, the two approaches are equivalent, but working with the augmented matrix requires less writing yet is easier to read, with some practice. Instead of dividing an *equation* by a scalar,[7] you can divide a *row* by a scalar. Instead of adding a multiple of an equation to another equation, you can add a multiple of a row to another row.

As you perform elimination on the augmented matrix, you should always remember the linear system lurking behind the matrix. To illustrate this method, we perform the elimination both on the augmented matrix and on the linear system it represents:

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \begin{array}{l} \div 2 \\ \\ \\ \end{array} \qquad \left|\begin{array}{rrrr} 2x + & 8y + & 4z = & 2 \\ 2x + & 5y + & z = & 5 \\ 4x + & 10y - & z = & 1 \end{array}\right| \begin{array}{l} \div 2 \\ \\ \\ \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right] \begin{array}{l} \\ -2\,(\mathrm{I}) \\ -4\,(\mathrm{I}) \end{array} \qquad \left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ 2x + & 5y + & z = & 5 \\ 4x + & 10y - & z = & 1 \end{array}\right| \begin{array}{l} \\ -2\,(\mathrm{I}) \\ -4\,(\mathrm{I}) \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{array}\right] \begin{array}{l} \\ \div(-3) \\ \\ \end{array} \qquad \left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ & -3y - & 3z = & 3 \\ & -6y - & 9z = & -3 \end{array}\right| \begin{array}{l} \\ \div(-3) \\ \\ \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -6 & -9 & -3 \end{array}\right] \begin{array}{l} -4\,(\mathrm{II}) \\ \\ +6\,(\mathrm{II}) \end{array} \qquad \left|\begin{array}{rrrr} x + & 4y + & 2z = & 1 \\ & y + & z = & -1 \\ & -6y - & 9z = & -3 \end{array}\right| \begin{array}{l} -4\,(\mathrm{II}) \\ \\ +6\,(\mathrm{II}) \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -9 \end{array}\right] \begin{array}{l} \\ \\ \div(-3) \end{array} \qquad \left|\begin{array}{rrrr} x & - & 2z = & 5 \\ & y + & z = & -1 \\ & & -3z = & -9 \end{array}\right| \begin{array}{l} \\ \\ \div(-3) \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} +2\,(\mathrm{III}) \\ -\,(\mathrm{III}) \\ \\ \end{array} \qquad \left|\begin{array}{rrrr} x & - & 2z = & 5 \\ & y + & z = & -1 \\ & & z = & 3 \end{array}\right| \begin{array}{l} +2\,(\mathrm{III}) \\ -\,(\mathrm{III}) \\ \\ \end{array}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \left|\begin{array}{rrrr} x & & = & 11 \\ & y & = & -4 \\ & & z = & 3 \end{array}\right|.$$

The solution is often represented as a vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}.$$

Thus far we have been focusing on systems of 3 linear equations with 3 unknowns. Next we will develop a technique for solving systems of linear equations of arbitrary size.

[7] In vector and matrix algebra, the term *scalar* is synonymous with (real) number.

Here is an example of a system of three linear equations with five unknowns:

$$\left| \begin{array}{ccccc} x_1 - x_2 & & & + 4x_5 = 2 \\ & x_3 & & - \ x_5 = 2 \\ & & x_4 & - \ x_5 = 3 \end{array} \right|$$

We can proceed as in the example on page 4. We solve each equation for the leading variable:

$$\left| \begin{array}{l} x_1 = 2 + x_2 - 4x_5 \\ x_3 = 2 \qquad\quad + \ x_5 \\ x_4 = 3 \qquad\quad + \ x_5 \end{array} \right|.$$

Now we can freely choose values for the nonleading variables, $x_2 = t$ and $x_5 = r$, for example. The leading variables are then determined by these choices:

$$x_1 = 2 + t - 4r, \qquad x_3 = 2 + r, \qquad x_4 = 3 + r.$$

This system has infinitely many solutions; we can write the solutions in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 & +t & -4r \\ & t & \\ 2 & & +r \\ 3 & & +r \\ & & r \end{bmatrix}.$$

Again, you can check this answer by substituting the solutions into the original equations, for example, $x_3 - x_5 = (2 + r) - r = 2$.

What makes this system so easy to solve? The following three properties are responsible for the simplicity of the solution, with the second property playing a key role:

- P1: The leading coefficient in each equation is 1. (The leading coefficient is the coefficient of the leading variable.)
- P2: The leading variable in each equation does not appear in any of the other equations. (For example, the leading variable x_3 of the second equation appears neither in the first nor in the third equation.)
- P3: The leading variables appear in the "natural order," with increasing indices as we go down the system (x_1, x_3, x_4 as opposed to x_3, x_1, x_4, for example).

Whenever we encounter a linear system with these three properties, we can solve for the leading variables and then choose arbitrary values for the other, nonleading variables, as we did above and on page 4.

Now we are ready to tackle the case of an arbitrary system of linear equations. We will illustrate our approach by means of an example:

$$\left| \begin{array}{rrrrrl} 2x_1 + & 4x_2 - & 2x_3 + & 2x_4 + & 4x_5 = & 2 \\ x_1 + & 2x_2 - & x_3 + & 2x_4 & = & 4 \\ 3x_1 + & 6x_2 - & 2x_3 + & x_4 + & 9x_5 = & 1 \\ 5x_1 + & 10x_2 - & 4x_3 + & 5x_4 + & 9x_5 = & 9 \end{array} \right|.$$

We wish to reduce this system to a system satisfying the three properties (P1, P2, and P3); this reduced system will then be easy to solve.

We will proceed from equation to equation, from top to bottom. The leading variable in the first equation is x_1, with leading coefficient 2. To satisfy property P1, we will divide this equation by 2. To satisfy property P2 for the variable x_1, we will then subtract suitable multiples of the first equation from the other three equations to eliminate the variable x_1 from those equations. We will perform these operations both on the system and on the augmented matrix.

$$
\left|\begin{array}{rcl} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 &=& 2 \\ x_1 + 2x_2 - x_3 + 2x_4 &=& 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 &=& 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 &=& 9 \end{array}\right| \div 2 \qquad \left[\begin{array}{rrrrr|r} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \div 2
$$

$$\downarrow$$

$$
\left|\begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 + 2x_5 &=& 1 \\ x_1 + 2x_2 - x_3 + 2x_4 &=& 4 \\ 3x_1 + 6x_2 - 2x_3 + x_4 + 9x_5 &=& 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 &=& 9 \end{array}\right| \begin{array}{l} \\ -(I) \\ -3(I) \\ -5(I) \end{array} \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \begin{array}{l} \\ -(I) \\ -3(I) \\ -5(I) \end{array}
$$

$$\downarrow$$

$$
\left|\begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 + 2x_5 &=& 1 \\ x_4 - 2x_5 &=& 3 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_3 - x_5 &=& 4 \end{array}\right| \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right]
$$

Now on to the second equation, with leading variable x_4 and leading coefficient 1. We could eliminate x_4 from the first and third equations and then proceed to the third equation, with leading variable x_3. However, this approach would violate our requirement P3 that the variables must be listed in the natural order, with increasing indices as we go down the system. To satisfy this requirement, we will swap the second equation with the third equation. (In the following summary, we will specify when such a swap is indicated and how it is to be performed.)

Then we can eliminate x_3 from the first and fourth equations.

$$
\left|\begin{array}{rcl} x_1 + 2x_2 - x_3 + x_4 + 2x_5 &=& 1 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ x_3 - x_5 &=& 4 \end{array}\right| \begin{array}{l} +(II) \\ \\ \\ -(II) \end{array} \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right] \begin{array}{l} +(II) \\ \\ \\ -(II) \end{array}
$$

$$\downarrow$$

$$
\left|\begin{array}{rcl} x_1 + 2x_2 - x_4 + 5x_5 &=& -1 \\ x_3 - 2x_4 + 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ 2x_4 - 4x_5 &=& 6 \end{array}\right| \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right]
$$

Now we turn our attention to the third equation, with leading variable x_4. We need to eliminate x_4 from the other three equations.

$$
\left|\begin{array}{rcl} x_1 + 2x_2 - x_4 + 5x_5 &=& -1 \\ x_3 - 2x_4 \quad 3x_5 &=& -2 \\ x_4 - 2x_5 &=& 3 \\ 2x_4 - 4x_5 &=& 6 \end{array}\right| \begin{array}{l} +(III) \\ +2(III) \\ \\ -2(III) \end{array} \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right] \begin{array}{l} +(III) \\ +2(III) \\ \\ -2(III) \end{array}
$$

$$\downarrow$$

$$
\left|\begin{array}{rcl} x_1 + 2x_2 + 3x_5 &=& 2 \\ x_3 - x_5 &=& 4 \\ x_4 - 2x_5 &=& 3 \\ 0 &=& 0 \end{array}\right| \qquad \left[\begin{array}{rrrrr|r} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Since there are no variables left in the fourth equation, we are done. Our system now satisfies properties P1, P2, and P3. We can solve the equations for the leading variables:

$$\left| \begin{array}{l} x_1 = 2 - 2x_2 - 3x_5 \\ x_3 = 4 \qquad\quad\; + \;\; x_5 \\ x_4 = 3 \qquad\quad\; + 2x_5 \end{array} \right|$$

If we let $x_2 = t$ and $x_5 = r$, then the infinitely many solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 & -2t & -3r \\ & t & \\ 4 & & +r \\ 3 & & +2r \\ & & r \end{bmatrix}.$$

Let us summarize.

Solving a system of linear equations

We proceed from equation to equation, from top to bottom.

Suppose we get to the ith equation. Let x_j be the leading variable of the system consisting of the ith and all the subsequent equations. (If no variables are left in this system, then the process comes to an end.)

- If x_j does not appear in the ith equation, swap the ith equation with the first equation below that does contain x_j.
- Suppose the coefficient of x_j in the ith equation is c; thus this equation is of the form $cx_j + \cdots = \cdots$. Divide the ith equation by c.
- Eliminate x_j from all the other equations, above and below the ith, by subtracting suitable multiples of the ith equation from the others.

Now proceed to the next equation.

If an equation *zero* = *nonzero* emerges in this process, then the system fails to have solutions; the system is *inconsistent*.

When you are through without encountering an inconsistency, solve each equation for its leading variable. You may choose the nonleading variables freely; the leading variables are then determined by these choices.

This process can be performed on the augmented matrix. As you do so, just imagine the linear system lurking behind it.

In the preceding example, we reduced the augmented matrix

$$M = \left[\begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \quad \text{to} \quad E = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We say that the final matrix E is in reduced row-echelon form (rref).

Reduced row-echelon form

A matrix is in reduced row-echelon form if it satisfies all of the following conditions:

a. If a row has nonzero entries, then the first nonzero entry is a 1, called the *leading* 1 (or *pivot*) in this row.

b. If a column contains a leading 1, then all the other entries in that column are 0.

c. If a row contains a leading 1, then each row above it contains a leading 1 further to the left.

Condition c implies that rows of 0's, if any, appear at the bottom of the matrix.

Conditions a, b, and c defining the reduced row-echelon form correspond to the conditions P1, P2, and P3 that we imposed on the system.

Note that the leading 1's in the matrix

$$E = \left[\begin{array}{ccccc|c} \enclose{circle}{1} & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & \enclose{circle}{1} & 0 & -1 & 4 \\ 0 & 0 & 0 & \enclose{circle}{1} & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

correspond to the leading variables in the reduced system,

$$\left| \begin{array}{rcrcrcl} \enclose{circle}{x_1} + 2x_2 & & & & + 3x_5 & = & 2 \\ & & \enclose{circle}{x_3} & & - x_5 & = & 4 \\ & & & \enclose{circle}{x_4} & - 2x_5 & = & 3 \end{array} \right| .$$

Here we draw the staircase formed by the leading variables. This is where the name *echelon form* comes from. According to Webster, an echelon is a formation "like a series of steps."

The operations we perform when bringing a matrix into reduced row-echelon form are referred to as elementary row operations. Let's review the three types of such operations.

Types of elementary row operations

- Divide a row by a nonzero scalar.
- Subtract a multiple of a row from another row.
- Swap two rows.

Consider the following system:

$$\left| \begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & & & - & 5x_4 & = & -7 \\ 3x_1 & - & 12x_2 & - & 2x_3 & - & 27x_4 & = & -33 \\ -2x_1 & + & 10x_2 & + & 2x_3 & + & 24x_4 & = & 29 \\ -x_1 & + & 6x_2 & + & x_3 & + & 14x_4 & = & 17 \end{array} \right| .$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -3 & 0 & -5 & -7 \\ 3 & -12 & -2 & -27 & -33 \\ -2 & 10 & 2 & 24 & 29 \\ -1 & 6 & 1 & 14 & 17 \end{array}\right].$$

The reduced row-echelon form for this matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

(We leave it to you to perform the elimination.)

Since the last row of the echelon form represents the equation $0 = 1$, the system is inconsistent.

This method of solving linear systems is sometimes referred to as *Gauss–Jordan elimination*, after the German mathematician Carl Friedrich Gauss (1777–1855; see Figure 4), perhaps the greatest mathematician of modern times, and the German engineer Wilhelm Jordan (1844–1899). Gauss himself called the method *eliminatio vulgaris*. Recall that the Chinese were using this method 2,000 years ago.

Figure 4 Carl Friedrich Gauss appears on an old German 10-mark note. (In fact, this is the *mirror image* of a well-known portrait of Gauss.[8])

How Gauss developed this method is noteworthy. On January 1, 1801, the Sicilian astronomer Giuseppe Piazzi (1746–1826) discovered a planet, which he named Ceres, in honor of the patron goddess of Sicily. Today, Ceres is called a dwarf planet, because it is only about 1,000 kilometers in diameter. Piazzi was able to observe Ceres for 40 nights, but then he lost track of it. Gauss, however, at the age of 24, succeeded in calculating the orbit of Ceres, even though the task seemed hopeless on the basis of a few observations. His computations were so accurate that the German astronomer W. Olbers (1758–1840) located the asteroid on December 31, 1801. In the course of his computations, Gauss had to solve systems of 17 linear equations.[9] In dealing with this problem, Gauss also used the method of least

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[8]Reproduced by permission of the German Bundesbank.

[9]For the mathematical details, see D. Teets and K. Whitehead, "The Discovery of Ceres: How Gauss Became Famous," *Mathematics Magazine*, 72, 2 (April 1999): 83–93.

squares, which he had developed around 1794. (See Section 5.4.) Since Gauss at first refused to reveal the methods that led to this amazing accomplishment, some even accused him of sorcery. Gauss later described his methods of orbit computation in his book *Theoria Motus Corporum Coelestium* (1809).

The method of solving a linear system by Gauss–Jordan elimination is called an *algorithm*.[10] An algorithm can be defined as "a finite procedure, written in a fixed symbolic vocabulary, governed by precise instructions, moving in discrete Steps, 1, 2, 3, . . . , whose execution requires no insight, cleverness, intuition, intelligence, or perspicuity, and that sooner or later comes to an end" (David Berlinski, *The Advent of the Algorithm: The Idea that Rules the World*, Harcourt Inc., 2000).

Gauss–Jordan elimination is well suited for solving linear systems on a computer, at least in principle. In practice, however, some tricky problems associated with roundoff errors can occur.

Numerical analysts tell us that we can reduce the proliferation of roundoff errors by modifying Gauss–Jordan elimination, employing more sophisticated reduction techniques.

In modifying Gauss–Jordan elimination, an interesting question arises: If we transform a matrix A into a matrix B by a sequence of elementary row operations and if B is in reduced row-echelon form, is it necessarily true that $B = \text{rref}(A)$? Fortunately (and perhaps surprisingly) this is indeed the case.

In this text, we will not utilize this fact, so there is no need to present the somewhat technical proof. If you feel ambitious, try to work out the proof yourself after studying Chapter 3. (See Exercises 3.3.84 through 3.3.87.)

[10] The word *algorithm* is derived from the name of the mathematician al-Khowarizmi, who introduced the term *algebra* into mathematics. (See page 1.)

EXERCISES 1.2

GOAL *Use Gauss–Jordan elimination to solve linear systems. Do simple problems using paper and pencil, and use technology to solve more complicated problems.*

In Exercises 1 through 12, find all solutions of the equations with paper and pencil using Gauss–Jordan elimination. Show all your work. Solve the system in Exercise 8 for the variables x_1, x_2, x_3, x_4, *and* x_5.

1. $\left| \begin{array}{l} x + y - 2z = 5 \\ 2x + 3y + 4z = 2 \end{array} \right|$

2. $\left| \begin{array}{l} 3x + 4y - z = 8 \\ 6x + 8y - 2z = 3 \end{array} \right|$

3. $x + 2y + 3z = 4$

4. $\left| \begin{array}{l} x + y = 1 \\ 2x - y = 5 \\ 3x + 4y = 2 \end{array} \right|$

5. $\left| \begin{array}{l} x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_4 = 0 \end{array} \right|$

6. $\left| \begin{array}{l} x_1 - 7x_2 + x_5 = 3 \\ x_3 - 2x_5 = 2 \\ x_4 + x_5 = 1 \end{array} \right|$

7. $\left| \begin{array}{l} x_1 + 2x_2 \quad 2x_4 + 3x_5 = 0 \\ x_3 + 3x_4 + 2x_5 = 0 \\ x_3 + 4x_4 - x_5 = 0 \\ x_5 = 0 \end{array} \right|$

8. $\left| \begin{array}{l} x_2 + 2x_4 + 3x_5 = 0 \\ 4x_4 + 8x_5 = 0 \end{array} \right|$

9. $\left| \begin{array}{l} x_4 + 2x_5 - x_6 = 2 \\ x_1 + 2x_2 + x_5 - x_6 = 0 \\ x_1 + 2x_2 + 2x_3 - x_5 + x_6 = 2 \end{array} \right|$

10. $\left| \begin{array}{l} 4x_1 + 3x_2 + 2x_3 - x_4 = 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 = 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 = -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 = 11 \end{array} \right|$

11. $\left| \begin{array}{l} x_1 + 2x_3 + 4x_4 = -8 \\ x_2 - 3x_3 - x_4 = 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 = 0 \\ -x_2 + 3x_3 + 4x_4 = -12 \end{array} \right|$

12. $\begin{vmatrix} 2x_1 & & -3x_3 & & +7x_5+ & 7x_6=0 \\ -2x_1+ & x_2+6x_3 & & & -6x_5- & 12x_6=0 \\ & x_2-3x_3 & & & +\ x_5+ & 5x_6=0 \\ & -2x_2 & & +x_4+ & x_5+ & x_6=0 \\ 2x_1+ & x_2-3x_3 & & & +8x_5+ & 7x_6=0 \end{vmatrix}$

Solve the linear systems in Exercises 13 through 17. You may use technology.

13. $\begin{vmatrix} 3x+11y+19z=-2 \\ 7x+23y+39z=10 \\ -4x-3y-2z=6 \end{vmatrix}$

14. $\begin{vmatrix} 3x+6y+14z=22 \\ 7x+14y+30z=46 \\ 4x+8y+7z=6 \end{vmatrix}$

15. $\begin{vmatrix} 3x+5y+3z=25 \\ 7x+9y+19z=65 \\ -4x+5y+11z=5 \end{vmatrix}$

16. $\begin{vmatrix} 3x_1+6x_2+9x_3+5x_4+25x_5=53 \\ 7x_1+14x_2+21x_3+9x_4+53x_5=105 \\ -4x_1-8x_2-12x_3+5x_4-10x_5=11 \end{vmatrix}$

17. $\begin{vmatrix} 2x_1+4x_2+3x_3+5x_4+6x_5=37 \\ 4x_1+8x_2+7x_3+5x_4+2x_5=74 \\ -2x_1-4x_2+3x_3+4x_4-5x_5=20 \\ x_1+2x_2+2x_3-x_4+2x_5=26 \\ 5x_1-10x_2+4x_3+6x_4+4x_5=24 \end{vmatrix}$

18. Determine which of the matrices below are in reduced row-echelon form:

a. $\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$

19. Find all 4×1 matrices in reduced row-echelon form.

20. We say that two $n \times m$ matrices in reduced row-echelon form are of the same type if they contain the same number of leading 1's in the same positions. For example,

$$\begin{bmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \textcircled{1} & 3 & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

are of the same type. How many types of 2×2 matrices in reduced row-echelon form are there?

21. How many types of 3×2 matrices in reduced row-echelon form are there? (See Exercise 20.)

22. How many types of 2×3 matrices in reduced row-echelon form are there? (See Exercise 20.)

23. Suppose you apply Gauss–Jordan elimination to a matrix. Explain how you can be sure that the resulting matrix is in reduced row-echelon form.

24. Suppose matrix A is transformed into matrix B by means of an elementary row operation. Is there an elementary row operation that transforms B into A? Explain.

25. Suppose matrix A is transformed into matrix B by a sequence of elementary row operations. Is there a sequence of elementary row operations that transforms B into A? Explain your answer. (See Exercise 24.)

26. Consider an $n \times m$ matrix A. Can you transform rref(A) into A by a sequence of elementary row operations? (See Exercise 25.)

27. Is there a sequence of elementary row operations that transforms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{into} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}?$$

Explain.

28. Suppose you subtract a multiple of an equation in a system from another equation in the system. Explain why the two systems (before and after this operation) have the same solutions.

29. *Balancing a chemical reaction.* Consider the chemical reaction

$$a\ NO_2 + b\ H_2O \rightarrow c\ HNO_2 + d\ HNO_3,$$

where a, b, c, and d are unknown positive integers. The reaction must be balanced; that is, the number of atoms of each element must be the same before and after the reaction. For example, because the number of oxygen atoms must remain the same,

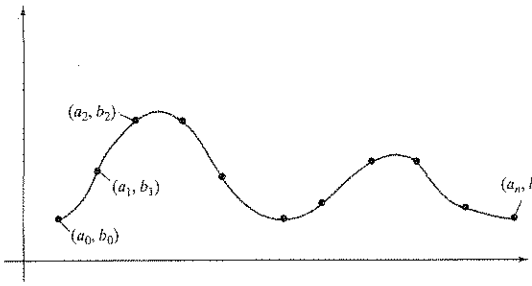
$$2a + b = 2c + 3d.$$

While there are many possible values for a, b, c, and d that balance the reaction, it is customary to use the smallest possible positive integers. Balance this reaction.

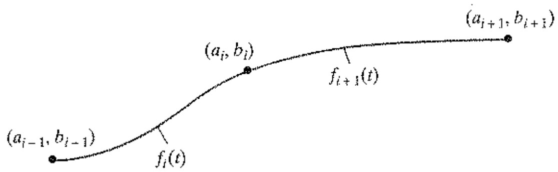
30. Find the polynomial of degree 3 [a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$] whose graph goes through the points $(0, 1)$, $(1, 0)$, $(-1, 0)$, and $(2, -15)$. Sketch the graph of this cubic.

31. Find the polynomial of degree 4 whose graph goes through the points $(1, 1)$, $(2, -1)$, $(3, -59)$, $(-1, 5)$, and $(-2, -29)$. Graph this polynomial.

32. *Cubic splines.* Suppose you are in charge of the design of a roller coaster ride. This simple ride will not make any left or right turns; that is, the track lies in a vertical plane. The accompanying figure shows the ride as viewed from the side. The points (a_i, b_i) are given to you, and your job is to connect the dots in a reasonably smooth way. Let $a_{i+1} > a_i$.



One method often employed in such design problems is the technique of cubic splines. We choose $f_i(t)$, a polynomial of degree ≤ 3, to define the shape of the ride between (a_{i-1}, b_{i-1}) and (a_i, b_i), for $i = 1, \ldots, n$.



Obviously, it is required that $f_i(a_i) = b_i$ and $f_i(a_{i-1}) = b_{i-1}$, for $i = 1, \ldots, n$. To guarantee a smooth ride at the points (a_i, b_i), we want the first and the second derivatives of f_i and f_{i+1} to agree at these points:

$$\begin{aligned} f_i'(a_i) &= f_{i+1}'(a_i) && \text{and} \\ f_i''(a_i) &= f_{i+1}''(a_i), && \text{for } i = 1, \ldots, n-1. \end{aligned}$$

Explain the practical significance of these conditions. Explain why, for the convenience of the riders, it is also required that

$$f_1'(a_0) = f_n'(a_n) = 0.$$

Show that satisfying all these conditions amounts to solving a system of linear equations. How many variables are in this system? How many equations? (*Note*: It can be shown that this system has a unique solution.)

33. Find the polynomial $f(t)$ of degree 3 such that $f(1) = 1$, $f(2) = 5$, $f'(1) = 2$, and $f'(2) = 9$, where $f'(t)$ is the derivative of $f(t)$. Graph this polynomial.

34. The *dot product* of two vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$ is defined by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Note that the dot product of two vectors is a scalar. We say that the vectors $\vec{x}$ and $\vec{y}$ are *perpendicular* if $\vec{x} \cdot \vec{y} = 0$.

Find all vectors in $\mathbb{R}^3$ perpendicular to

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

Draw a sketch.

35. Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}.$$

(See Exercise 34.)

36. Find all solutions x_1, x_2, x_3 of the equation

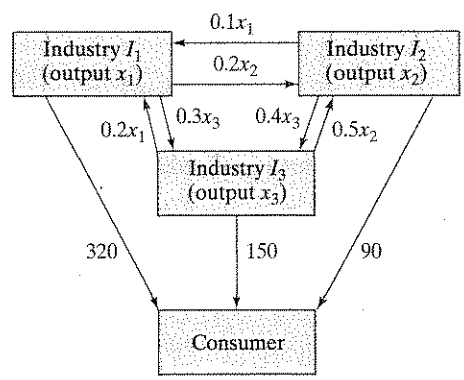
$$\vec{b} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3,$$

where

$$\vec{b} = \begin{bmatrix} -8 \\ -1 \\ 2 \\ 15 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 5 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 1 \end{bmatrix}.$$

37. For some background on this exercise, see Exercise 1.1.20.

Consider an economy with three industries, I_1, I_2, I_3. What outputs x_1, x_2, x_3 should they produce to satisfy both consumer demand and interindustry demand? The demands put on the three industries are shown in the accompanying figure.



38. If we consider more than three industries in an input–output model, it is cumbersome to represent all the demands in a diagram as in Exercise 37. Suppose we have the industries $I_1, I_2, \ldots, I_n$, with outputs $x_1, x_2, \ldots, x_n$. The *output vector* is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The *consumer demand vector* is

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where b_i is the consumer demand on industry I_i. The demand vector for industry I_j is

$$\vec{v}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix},$$

where a_{ij} is the demand industry I_j puts on industry I_i, for each \$1 of output industry I_j produces. For example, $a_{32} = 0.5$ means that industry I_2 needs 50¢worth of products from industry I_3 for each \$1 worth of goods I_2 produces. The coefficient a_{ii} need not be 0: Producing a product may require goods or services from the same industry.

a. Find the four demand vectors for the economy in Exercise 37.

b. What is the meaning in economic terms of $x_j\vec{v}_j$?

c. What is the meaning in economic terms of $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b}$?

d. What is the meaning in economic terms of the equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n + \vec{b} = \vec{x}?$$

39. Consider the economy of Israel in 1958.[11] The three industries considered here are

I_1 : agriculture,
I_2 : manufacturing,
I_3 : energy.

Outputs and demands are measured in millions of Israeli pounds, the currency of Israel at that time. We are told that

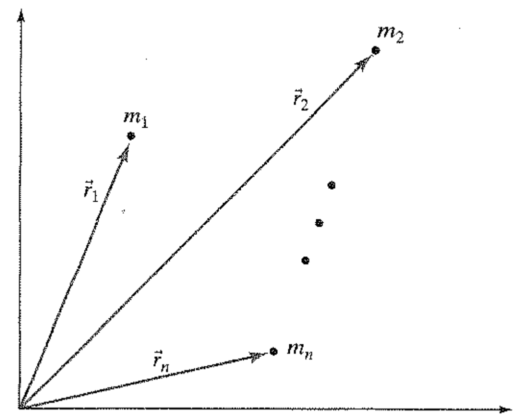
$$\vec{b} = \begin{bmatrix} 13.2 \\ 17.6 \\ 1.8 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 0.293 \\ 0.014 \\ 0.044 \end{bmatrix},$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0.207 \\ 0.01 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0.017 \\ 0.216 \end{bmatrix}.$$

a. Why do the first components of $\vec{v}_2$ and $\vec{v}_3$ equal 0?

b. Find the outputs x_1, x_2, x_3 required to satisfy demand.

40. Consider some particles in the plane with position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ and masses $m_1, m_2, \ldots, m_n$.

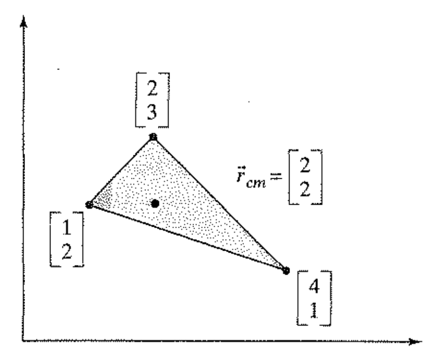


The position vector of the *center of mass* of this system is

$$\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n),$$

where $M = m_1 + m_2 + \cdots + m_n$.

Consider the triangular plate shown in the accompanying sketch. How must a total mass of 1 kg be distributed among the three vertices of the plate so that the plate can be supported at the point $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$; that is, $\vec{r}_{cm} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? Assume that the mass of the plate itself is negligible.



41. The *momentum* $\vec{P}$ of a system of n particles in space with masses $m_1, m_2, \ldots, m_n$ and velocities $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is defined as

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_n\vec{v}_n.$$

Now consider two elementary particles with velocities

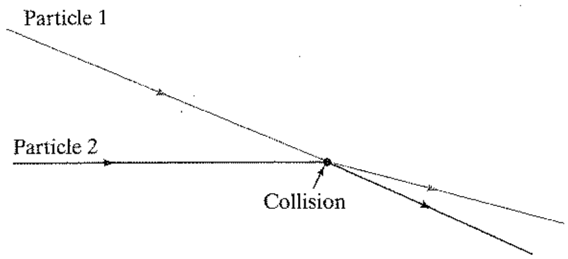
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix}.$$

[11] W. Leontief, *Input–Output Economics*, Oxford University Press, 1966.

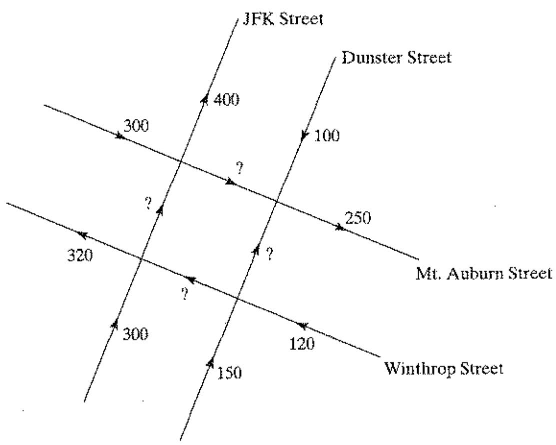
The particles collide. After the collision, their respective velocities are observed to be

$$\vec{w}_1 = \begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}.$$

Assume that the momentum of the system is conserved throughout the collision. What does this experiment tell you about the masses of the two particles? (See the accompanying figure.)



42. The accompanying sketch represents a maze of one-way streets in a city in the United States. The traffic volume through certain blocks during an hour has been measured. Suppose that the vehicles leaving the area during this hour were exactly the same as those entering it.



What can you say about the traffic volume at the four locations indicated by a question mark? Can you figure out exactly how much traffic there was on each block? If not, describe one possible scenario. For each of the four locations, find the highest and the lowest possible traffic volume.

43. Let $S(t)$ be the length of the tth day of the year 2009 in Mumbai (formerly known as Bombay), India (measured in hours, from sunrise to sunset). We are given the following values of $S(t)$:

t	$S(t)$
47	11.5
74	12
273	12

For example, $S(47) = 11.5$ means that the time from sunrise to sunset on February 16 is 11 hours and 30 minutes. For locations close to the equator, the function $S(t)$ is well approximated by a trigonometric function of the form

$$S(t) = a + b\cos\left(\frac{2\pi t}{365}\right) + c\sin\left(\frac{2\pi t}{365}\right).$$

(The period is 365 days, or 1 year.) Find this approximation for Mumbai, and graph your solution. According to this model, how long is the longest day of the year in Mumbai?

44. Kyle is getting some flowers for Olivia, his Valentine. Being of a precise analytical mind, he plans to spend exactly \$24 on a bunch of exactly two dozen flowers. At the flower market they have lilies (\$3 each), roses (\$2 each), and daisies (\$0.50 each). Kyle knows that Olivia loves lilies; what is he to do?

45. Consider the equations

$$\left| \begin{array}{rcl} x + 2y + & 3z & = 4 \\ x + ky + & 4z & = 6 \\ x + 2y + & (k+2)z & = 6 \end{array} \right|,$$

where k is an arbitrary constant.

a. For which values of the constant k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

46. Consider the equations

$$\left| \begin{array}{rcl} y + 2kz & = & 0 \\ x + 2y + 6z & = & 2 \\ kx \quad\quad + 2z & = & 1 \end{array} \right|,$$

where k is an arbitrary constant.

a. For which values of the constant k does this system have a unique solution?

b. When is there no solution?

c. When are there infinitely many solutions?

47. **a.** Find all solutions x_1, x_2, x_3, x_4 of the system $x_2 = \frac{1}{2}(x_1 + x_3)$, $x_3 = \frac{1}{2}(x_2 + x_4)$.

b. In part (a), is there a solution with $x_1 = 1$ and $x_4 = 13$?

48. For an arbitrary positive integer $n \geq 3$, find all solutions $x_1, x_2, x_3, \ldots, x_n$ of the simultaneous equations $x_2 = \frac{1}{2}(x_1+x_3), x_3 = \frac{1}{2}(x_2+x_4), \ldots, x_{n-1} = \frac{1}{2}(x_{n-2}+x_n)$. Note that we are asked to solve the simultaneous equations $x_k = \frac{1}{2}(x_{k-1} + x_{k+1})$, for $k = 2, 3, \ldots, n-1$.

49. Consider the system

$$\left| \begin{array}{rcl} 2x + y \quad\quad & = & C \\ 3y + z & = & C \\ x \quad\quad + 4z & = & C \end{array} \right|,$$

where C is a constant. Find the smallest positive integer C such that x, y, and z are all integers.

50. Find all the polynomials $f(t)$ of degree ≤ 3 such that $f(0) = 3$, $f(1) = 2$, $f(2) = 0$, and $\int_0^2 f(t)\,dt = 4$. (If you have studied Simpson's rule in calculus, explain the result.)

Exercises 51 through 60 are concerned with conics. A conic is a curve in $\mathbb{R}^2$ that can be described by an equation of the form $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 = 0$, where at least one of the coefficients c_i is nonzero. Examples are circles, ellipses, hyperbolas, and parabolas. If k is any nonzero constant, then the equations $f(x, y) = 0$ and $kf(x, y) = 0$ describe the same conic. For example, the equation $-4 + x^2 + y^2 = 0$ and $-12 + 3x^2 + 3y^2 = 0$ both describe the circle of radius 2 centered at the origin. In Exercises 51 through 60, find all the conics through the given points, and draw a rough sketch of your solution curve(s).

51. $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, and $(0, 2)$.

52. $(0, 0)$, $(2, 0)$, $(0, 2)$, $(2, 2)$, and $(1, 3)$.

53. $(0, 0)$, $(1, 0)$, $(2, 0)$, $(3, 0)$, and $(1, 1)$.

54. $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(1, 0)$.

55. $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$.

56. $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, -1)$.

57. $(5, 0)$, $(1, 2)$, $(2, 1)$, $(8, 1)$, and $(2, 9)$.

58. $(1, 0)$, $(2, 0)$, $(2, 2)$, $(5, 2)$, and $(5, 6)$.

59. $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, $(0, 2)$, and $(1, 1)$.

60. $(0, 0)$, $(2, 0)$, $(0, 2)$, $(2, 2)$, $(1, 3)$, and $(4, 1)$.

61. Students are buying books for the new semester. Eddie buys the environmental statistics book and the set theory book for $178. Leah, who is buying books for herself and her friend, spends $319 on two environmental statistics books, one set theory book, and one educational psychology book. Mehmet buys the educational psychology book and the set theory book for $147 in total. How much does each book cost?

62. Students are buying books for the new semester. Brigitte buys the German grammar book and the German novel, *Die Leiden des jungen Werther*, for €64 in total. Claude spends €98 on the linear algebra text and the German grammar book, while Denise buys the linear algebra text and *Werther*, for €76. How much does each of the three books cost?

63. At the beginning of a political science class at a large university, the students were asked which term, *liberal* or *conservative*, best described their political views. They were asked the same question at the end of the course, to see what effect the class discussions had on their views. Of those that characterized themselves as "liberal" initially, 30% held conservative views at the end. Of those who were conservative initially, 40% moved to the liberal camp. It turned out that there were just as many students with conservative views at the end as there had been liberal students at the beginning. Out of the 260 students in the class, how many held liberal and conservative views at the beginning of the course and at the end? (No students joined or dropped the class between the surveys, and they all participated in both surveys.)

64. At the beginning of a semester, 55 students have signed up for Linear Algebra; the course is offered in two sections that are taught at different times. Because of scheduling conflicts and personal preferences, 20% of the students in Section A switch to Section B in the first few weeks of class, while 30% of the students in Section B switch to A, resulting in a net loss of 4 students for Section B. How large were the two sections at the beginning of the semester? No students dropped Linear Algebra (why would they?) or joined the course late.

Historical Problems

65. Five cows and two sheep together cost ten *liang*[12] of silver. Two cows and five sheep together cost eight *liang* of silver. What is the cost of a cow and a sheep, respectively? (*Nine Chapters*,[13] Chapter 8, Problem 7)

66. If you sell two cows and five sheep and you buy 13 pigs, you gain 1,000 coins. If you sell three cows and three pigs and buy nine sheep, you break even. If you sell six sheep and eight pigs and you buy five cows, you lose 600 coins. What is the price of a cow, a sheep, and a pig, respectively? (*Nine Chapters*, Chapter 8, Problem 8)

67. You place five sparrows on one of the pans of a balance and six swallows on the other pan; it turns out that the sparrows are heavier. But if you exchange one sparrow and one swallow, the weights are exactly balanced. All the birds together weigh 1 *jin*. What is the weight of a sparrow and a swallow, respectively? [Give the answer in *liang*, with 1 *jin* = 16 *liang*.] (*Nine Chapters*, Chapter 8, Problem 9)

68. Consider the task of pulling a weight of 40 *dan*[14] up a hill; we have one military horse, two ordinary horses, and three weak horses at our disposal to get the job done. It turns out that the military horse and one of the ordinary horses, pulling together, are barely able to pull the

[12] A *liang* was about 16 grams at the time of the Han Dynasty.

[13] See page 1; we present some of the problems from the *Nine Chapters on the Mathematical Art* in a free translation, with some additional explanations, since the scenarios discussed in a few of these problems are rather unfamiliar to the modern reader.

[14] 1 *dan* = 120 *jin* = 1,920 *liang*. Thus a *dan* was about 30 kilograms at that time.

weight (but they could not pull any more). Likewise, the two ordinary horses together with one weak horse are just able to do the job, as are the three weak horses together with the military horse. How much weight can each of the horses pull alone? (*Nine Chapters*, Chapter 8, Problem 12)

69. Five households share a deep well for their water supply. Each household owns a few ropes of a certain length, which varies only from household to household. The five households, A, B, C, D, and E, own 2, 3, 4, 5, and 6 ropes, respectively. Even when tying all their ropes together, none of the households alone are able to reach the water, but A's two ropes together with one of B's ropes just reach the water. Likewise, B's three ropes with one of C's ropes, C's four ropes with one of D's ropes, D's five ropes with one of E's ropes, and E's six ropes with one of A's ropes all just reach the water. How long are the ropes of the various households, and how deep is the well?
Commentary: As stated, this problem leads to a system of 5 linear equations in 6 variables; with the given information, we are unable to determine the depth of the well. The *Nine Chapters* gives one particular solution, where the depth of the well is 7 *zhang*,[15] 2 *chi*, 1 *cun*, or 721 *cun* (since 1 *zhang* = 10 *chi* and 1 *chi* = 10 *cun*). Using this particular value for the depth of the well, find the lengths of the various ropes.

70. "A rooster is worth five coins, a hen three coins, and 3 chicks one coin. With 100 coins we buy 100 of them. How many roosters, hens, and chicks can we buy?" (From the *Mathematical Manual* by Zhang Qiujian, Chapter 3, Problem 38; 5th century A.D.)
Commentary: This famous *Hundred Fowl Problem* has reappeared in countless variations in Indian, Arabic, and European texts (see Exercises 71 through 74); it has remained popular to this day (see Exercise 44 of this section).

71. "Pigeons are sold at the rate of 5 for 3 panas, sarasabirds at the rate of 7 for 5 panas, swans at the rate of 9 for 7 panas, and peacocks at the rate of 3 for 9 panas. A man was told to bring 100 birds for 100 panas for the amusement of the King's son. What does he pay for each of the various kinds of birds that he buys?" (From the *Ganita-Sara-Sangraha* by Mahavira, India; 9th century A.D.) Find one solution to this problem.

72. "A duck costs four coins, five sparrows cost one coin, and a rooster costs one coin. Somebody buys 100 birds for 100 coins. How many birds of each kind can he buy?" (From the *Key to Arithmetic* by Al-Kashi; 15th century)

73. "A certain person buys sheep, goats, and hogs, to the number of 100, for 100 crowns; the sheep cost him $\frac{1}{2}$ a crown a-piece; the goats, $1\frac{1}{3}$ crown; and the hogs $3\frac{1}{2}$ crowns. How many had he of each?" (From the *Elements of Algebra* by Leonhard Euler, 1770)

74. "A gentleman has a household of 100 persons and orders that they be given 100 measures of grain. He directs that each man should receive three measures, each woman two measures, and each child half a measure. How many men, women, and children are there in this household?" We are told that there is at least one man, one woman, and one child. (From the *Problems for Quickening a Young Mind* by Alcuin [c. 732–804], the Abbot of St. Martins at Tours. Alcuin was a friend and tutor to Charlemagne and his family at Aachen.)

75. A father, when dying, gave to his sons 30 barrels, of which 10 were full of wine, 10 were half full, and the last 10 were empty. Divide the wine and flasks so that there will be equal division among the three sons of both wine and barrels. Find all the solutions of this problem. (From Alcuin)

76. "Make me a crown weighing 60 *minae*, mixing gold, bronze, tin, and wrought iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold, tin, bronze, and iron you must put in." (From the *Greek Anthology* by Metrodorus, 6th century A.D.)

77. Three merchants find a purse lying in the road. One merchant says "If I keep the purse, I shall have twice as much money as the two of you together." "Give me the purse and I shall have three times as much as the two of you together" said the second merchant. The third merchant said "I shall be much better off than either of you if I keep the purse, I shall have five times as much as the two of you together." If there are 60 coins (of equal value) in the purse, how much money does each merchant have? (From Mahavira)

78. 3 cows graze 1 field bare in 2 days,
7 cows graze 4 fields bare in 4 days, and
3 cows graze 2 fields bare in 5 days.
It is assumed that each field initially provides the same amount, x, of grass; that the daily growth, y, of the fields remains constant; and that all the cows eat the same amount, z, each day. (Quantities x, y, and z are measured by weight.) Find all the solutions of this problem. (This is a special case of a problem discussed by Isaac Newton in his *Arithmetica Universalis*, 1707.)

[15] 1 *zhang* was about 2.3 meters at that time.