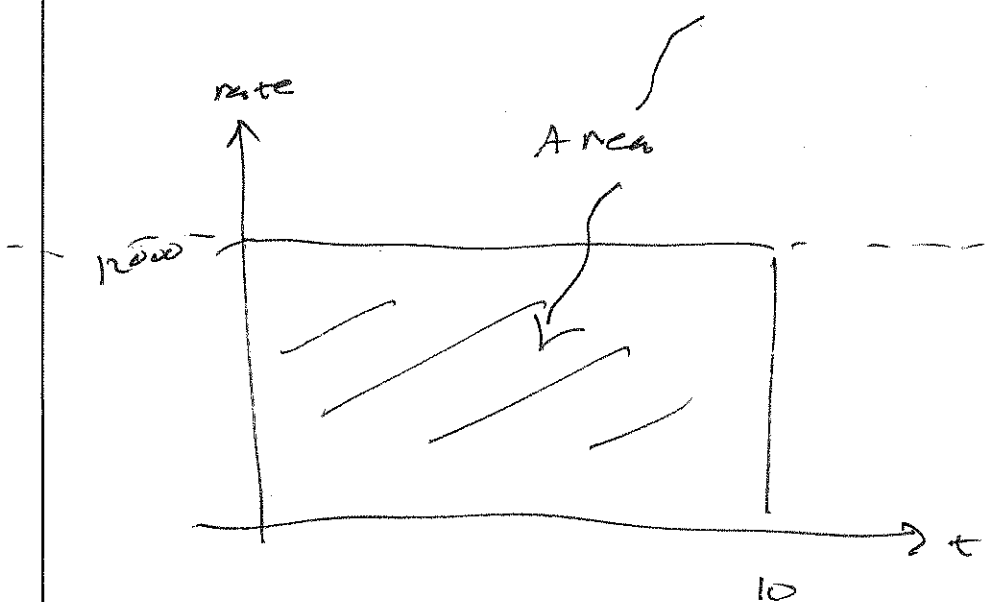
Section 13.04

Applications

Part 1: Continuous Income Streams

Example 1: Find the total income over the next 10 years from a continuous income stream that has an annual flow of $12,000 (imagine rent).

$10 \times 12000 = \$120,000$



Question: Is it reasonable to assume a continuous income stream?

In many cases - yes. A nice example is a vending machine.

Example 2: A small company models its monthly income with $f(t) = 10000\, e^{0.02\,t}$.
How much income can the company expect in its first two years (t in ~~years~~ months)?

$$\text{Income} = \int_0^{24} 10000\, e^{0.02t}\, dt$$

$$= \frac{10000}{0.02}\left[e^{0.02t}\right]_0^{24}$$

$$= \frac{10000}{0.02}\left[e^{0.48} - e^{0}\right]$$

$$= \$308{,}037$$

NOTE

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$$

$$\int e^{\frac{1}{3}x}\,dx = 3e^{\frac{1}{3}x} + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

Part 2: Present and Future Value

Present vs. future value

PV - what you need to know to buy a company.

FV - What you need to know if you own the company.

Let $f(t)$ be the rate of continuous income flow for k years earning interest at a rate r, compounded continuously. Then, the present value of the continuous income stream is:

$$\text{PV} = \int_0^k f(t)\, e^{-r\,t}\, d\,t$$

Recall $\underset{\uparrow}{A} = Pe^{rt} \longrightarrow \underset{\uparrow}{P} = Ae^{-rt}$.

Future value. Present value.

Example 3: A continuous income stream has an annual rate of flow of $f(t) = 9000\, e^{0.12\,t}$. Find the present value of this income stream for the next 10 years, if money is worth 6%, compounded continuously.

$$\text{PV} = \int_0^{10} 9000\, e^{0.12t}\, e^{-0.06t}\, dt$$

$$= \int_0^{10} 9000\, e^{0.06t}\, dt$$

$$= \frac{9000}{0.06}\left[e^{0.06t}\right]_0^{10}$$

$$= \frac{9000}{0.06}\left(e^{0.6} - e^{0}\right)$$

$$= \$123{,}318$$ (present value).

Let $f(t)$ be the rate of continuous income flow for k years earning interest at a rate r, compounded continuously. Then, the future value of the continuous income stream is:

$$\text{FV} = e^{rk} \underbrace{\int_0^k f(t)\, e^{-rt}\, dt}_{\text{PV - present value.}}$$

$$\text{FV} = \text{PV} \cdot e^{rk}$$

recall $A = Pe^{rt}$.

Example 4: Suppose that a continuous income stream has an annual rate of flow given by $f(t) = 5000\, e^{-0.01\, t}$ and suppose that money is worth 7% compounded continuously. Create the integral used to find:

a.) The total income for the next 5 years.

$$\text{PV} = 5000 \int_0^5 e^{-0.01t} \cdot e^{-.07t}\, dt$$

$$= 5000 \int_0^5 e^{-.08t}\, dt$$

$$= \frac{5000}{-.08} \left[e^{-.08t} \right]_0^5$$

$= -62500(e^{-.4} - 1)$

$= \$20{,}605$ (present value)

b.) The present value of the income stream for the next 5 years.

$$\text{Income} = 5000 \int_0^5 e^{-0.01t}\, dt$$

$$= \frac{5000}{-.01} \left[e^{-.01t} \right]_0^5$$

$$= -500000(e^{-.05} - 1)$$

Income $= \$24385$

I swapped (a) & (b)

c.) The future value of the income stream 5 years from today.

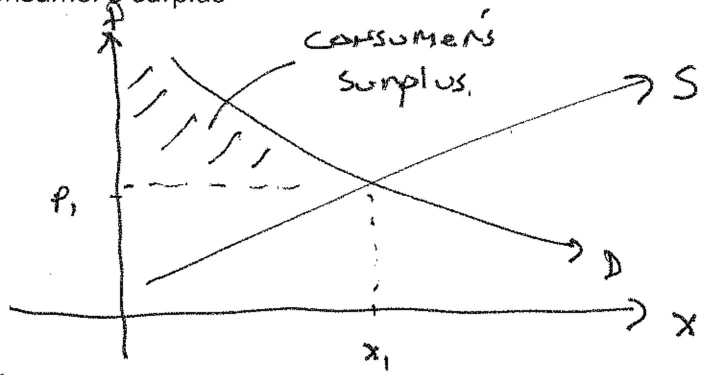
$$\text{FV} = \text{PV} \cdot e^{rk}$$

$$= 20{,}605\, e^{0.07(5)}$$

$$= \$29{,}240 \text{ (future value)}.$$

Part 3: Consumer and Supplier's Surplus

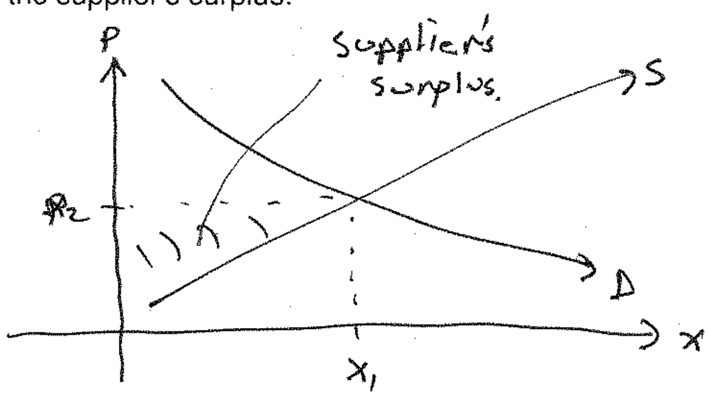
Consumer's Surplus:
Suppose that the demand for a product is given by $D: p = f(x)$ and that the supply of the product is described by $S: p = g(x)$. The price p_1 where the functions intersect is the equilibrium price. As the demand curve shows, some consumers would have been willing to pay more for the product that p_1; this is called the consumer's surplus



The formula:

$$CS = \int_0^{x_1} D\, dx - p_1\, x_1$$

Supplier's Surplus:
When a product is sold at the equilibrium price, some suppliers benefit as they would have been willing to sell at a lower price. We refer to this increased revenue as the supplier's surplus.



The formula:

$$SS = p_1\, x_1 - \int_0^{x_1} S\, dx$$

Example 5: If demand is $D: p = \frac{100}{x+1}$ and supply is $S: p = x + 1$, and market equilibrium is reached when 9 units are supplied at $10 each, create the integral used to find:

a.) The consumer's surplus

$$CS = \int_0^9 \frac{100}{x+1}\,dx - 9(10)$$

$$= 100\left[\ln(x+1)\right]_0^9 - 90$$

$$= 100(\ln(10) - \ln(1)) - 90$$

$$= \$140.$$

b.) The supplier's surplus

$$SS = 90 - \int_0^9 (x+1)\,dx$$

$$= 90 - \left[\frac{x^2}{2} + x\right]_0^9$$

$$= 90 - \left(\frac{81}{2} + 9\right)$$

$$= \$40.5$$