Section 13.02

The Definite Integral and the Fundamental Theorem of Calculus

Part 1: The Definite Integral

To find the area under $f(x)$ (perhaps it makes the most sense to assume $f \geq 0$) on the interval $[a, b]$, we evaluate:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

We formalize this quantity through the definition of the definite integral given below.

Δx rectangle width

x_i end point.

$f(x_i)$ rectangle height.

Definition: The definite integral
If f is continuous on the interval $[a, b]$, then the area under f is given by:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$ (the right end point of the i^{th} subinterval; each subinterval having equal width).

Example 1: Use the definition of the definite integral to write $\int_{-3}^{2}(4x-7)\,dx$ as the limit of sums.

$$\Delta x = \frac{2-(-3)}{n} = \frac{5}{n}$$

$$x_i = -3 + \frac{5}{n}i$$

$$f(x_i) = 4\left(-3+\frac{5}{n}i\right) - 7$$

$$\int_{-3}^{2}(4x-7)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[4\left(-3+\frac{5}{n}i\right)-7\right]\cdot\frac{5}{n}$$

Now, the notation for the indefinite integral should lead to the obvious question: what is the relationship between the indefinite integral and the definite?

Part 2: The Fundamental Theorem of Calculus

Definition: The Fundamental Theorem of Calculus
Let f be continuous on the interval $[a, b]$. Then, the definite integral exists and:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. That is, $F' = f$.

Example 2: $\int_0^1 x\,dx$ (Note: This involves finding the area of the triangle that we worked three ways in the previous section).

$$\int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

Example 3: $\int_0^1 x^3\, dx$

$$= \frac{x^4}{4}\Big|_0^1$$

$$= \frac{1}{4} - 0$$

$$= \frac{1}{4}$$

Example 4: $\int_1^9 \sqrt{x}\, dx$

$$= \int_1^9 x^{1/2}\, dx$$

$$= \frac{2}{3} x^{3/2}\Big|_1^9$$

$$= \frac{2}{3}(27-1)$$

$$= \frac{52}{3}$$

Example 5: $\int_0^5 4\sqrt[3]{x^2}\, dx$

$$= \int_0^5 4x^{2/3}\, dx$$

$$= \frac{12}{5} x^{5/3}\Big]_0^5$$

$$= \frac{12}{5} \cdot 5^{5/3}$$

$$= 12 \cdot 5^{2/3}$$

Example 6: $\int_2^4 (x^2+2)^3\, x\, dx \;=\; \frac{1}{8}(x^2+2)^4\Big]_2^4$

Let $u = x^2+2$ $\qquad = \frac{1}{8}(18^4 - 6^4)$

$du = 2x\,dx$

$\frac{1}{2}\int (x^2+2)^3 2x\,dx = \frac{1}{2}\int u^3\,du = \frac{1}{8}u^4 + c$

Example 7: $\int_{-1}^2 x \sqrt[3]{x^2-5}\, dx \;=\; \frac{3}{8}(x^2-5)^{4/3}\Big]_{-1}^2$

$= \frac{3}{8}\left[(-1)^{4/3} - (-4)^{4/3}\right]$

Let $u = x^2-5$

$du = 2x\,dx$

$\frac{1}{2}\int x\sqrt[3]{x^2-5}\,2\,dx = \frac{1}{2}\int \sqrt[3]{u}\,du$

$= \frac{1}{2}\int u^{1/3}\,du$

$= \frac{3}{8}u^{4/3} + c$

Example 8: Suppose that a vending machine service company models its income by assuming that money flows continuously into the machines with an annual rate of flow of $f(t) = 120\,e^{0.01\,t}$ where f gives the income in \$1,000/yr. Find the total income for the company over the first three years.

$$\text{Total income} = \int_0^3 120e^{0.01t}\,dt$$

$$= \frac{120}{0.01}e^{0.01t}\Big]_0^3$$

$$= 12{,}000(\underbrace{e^{0.03}}_{0.03}-1)$$

$$= 365.45$$

The total income is about \$365,000. over 3 years.