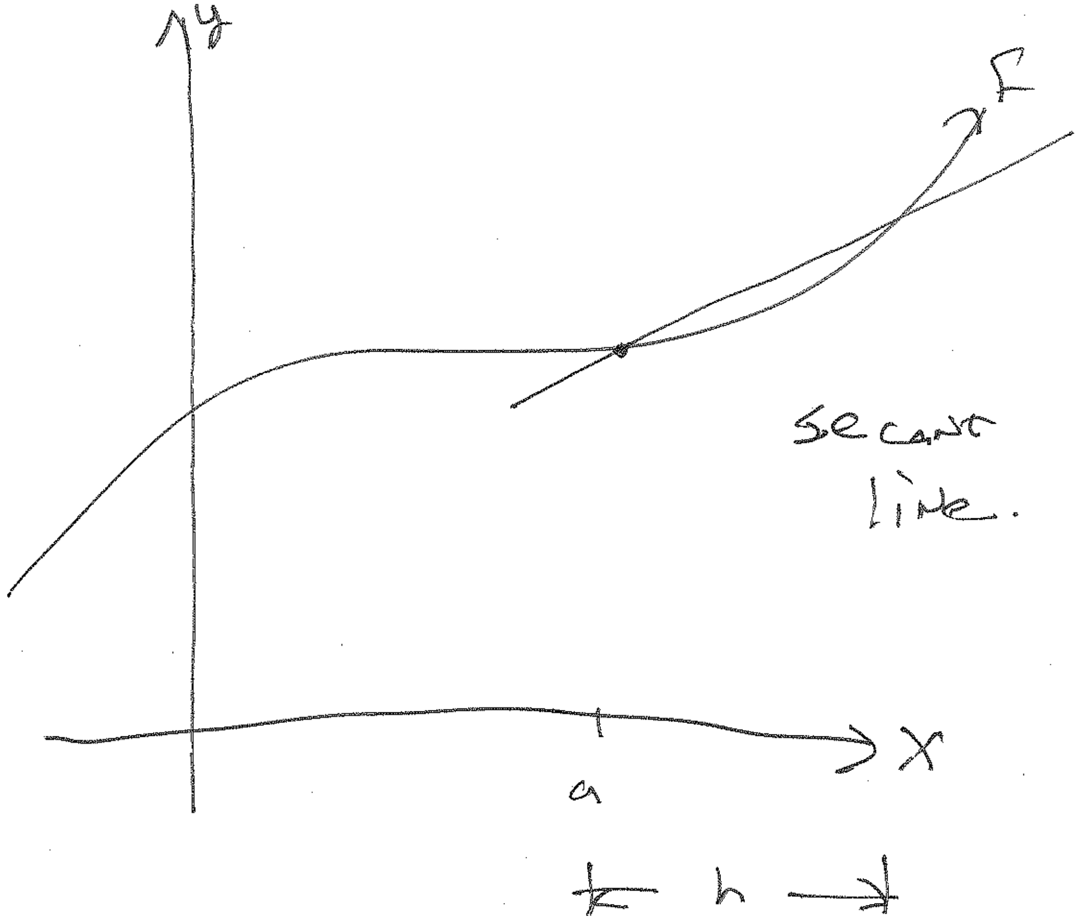
Section 9.3

Rates of Change: The Derivative

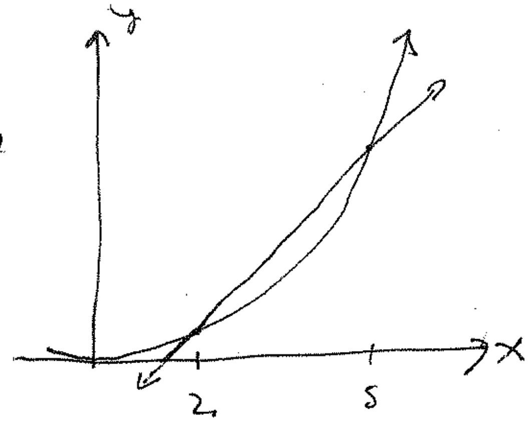
Part 1: Average Rates of Change (ROC)



Example 1: Find the average ROC of $f(x) = x^2$ on the intervals

a.) [2, 5]

$$\frac{25-4}{3} = \frac{21}{3} = 7$$



b.) [2, 4]

$$\frac{16-4}{2} = \frac{12}{2} = 6$$

c.) [2, 3]

$$\frac{9-4}{1} = \frac{5}{1} = 5$$

d.) [2, 2+h]

$$\frac{(2+h)^2 - 4}{h} = \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}$$

$$= 4 + h$$

Definition: Average Rate of Change (ROC)

The average ROC of $f(x)$ on $[a, b]$ provided that f is continuous on the interval is:

$$\frac{f(b) - f(a)}{b - a}$$

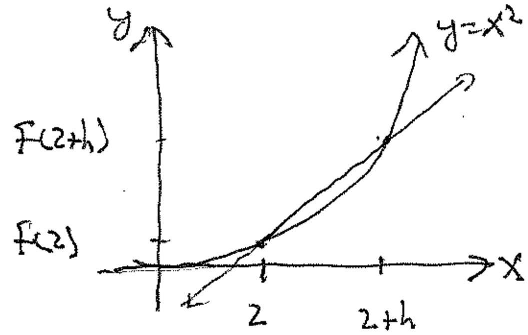
As the interval $[a, b]$ becomes smaller, (that is, as $(b - a) \to 0$), the average ROC approaches the instantaneous ROC.

Part 2: Instantaneous Rates of Change

Example 2: Find the instantaneous ROC of $f(x) = x^2$ at $x = 2$. Let h represent the change in x.

1st: Average ROC (and picture)

$$\frac{f(2+h) - f(2)}{h}$$



Definition: The Difference Quotient (I call it "DQ")

$$DQ = \frac{f(x+h)-f(x)}{h}$$

Example 2 continued:

2nd: Find $f(2+h) - f(2)$

$$(2+h)^2 - 2^2$$
$$= \cancel{4} + 4h + h^2 - \cancel{4}$$
$$= 4h + h^2$$

3rd: Form the difference quotient. (I have this as a seperate step because I find students struggle with step 2).

$$\frac{4h + h^2}{h} = 4 + h$$

4th: Find the instantaneous ROC of f at $x = 2$ by evaluating the limit of the DQ.

$$\lim_{h \to 0} (4 + h) = 4.$$

Part 3: The Derivative

Definition: The Derivative
If f is a function defined by $y = f(x)$, then the derivative of $f(x)$ at any value x, denoted $f'(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists. If $f'(c)$ exists, we say that f is differentiable at c.

Example 3: Find $f'(x)$ if $f(x) = 3x^2 - x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 - (x+h)] - [3x^2 - x]}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - x - h - 3x^2 + x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{x} - h - \cancel{3x^2} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - h}{h}$$

$$= \lim_{h \to 0} 6x + 3h - 1 = 6x - 1$$

Example 4 (for groups): Find $g'(x)$ for $g(x) =$ $3 - 2x$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{(3 - 2(x+h)) - (3 - 2x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} - \cancel{2x} - 2h - \cancel{3} + \cancel{2x}}{h}$$

$$= \lim_{h \to 0} -2$$

$$= -2$$

The derivative gives the slope at a point (remember, the average ROC gave the slope of a secant line and the instantaneous ROC will give the slope at a point).

For some problems, it can be beneficial to recall the point-slope form for the equation of a line with slope m and that includes the point (x_1, y_1). The formula is: $y - y_1 = m(x - x_1)$.

Example 5: Find the tangent line to $f(x) = 3x^2 - x$ when $x = -3$ (use the result of example 3).

$f'(x) = 6x - 1 \Rightarrow f'(-3) = -19$ is the slope of the tangent line thru $(-3, 30)$ and

$y - 30 = -19(x+3)$ is the eqt of the tangent line.

In chapter 1, marginal cost $\overline{\text{MC}}$ was defined as the slope of the cost function. marginal revenue and profit were similarly defined.

Example 6: Suppose the cost to produce x items is modeled by $C(x) = 25x + 1000$.

Review: What does the "+1000" represent in the context of the problem?

Fixed costs.

a.) Find the marginal cost.

$\overline{MC} = 25$ (slope)

b.) Interpret the result from part (a.).

For each additional item produced, cost increases by \$25.

Whereas in chapter 1, it was simple to find the slope because we need only determine the slope of a line, now we will find the slope at a point using calculus and the derivative.

Example 7: Suppose the profit from the sale of x cars (~~in $1,000's~~) is given by
$P(x) = 500x - x^2 - 100$

a.) Find $\overline{MP}$

$$P'(x) = \lim_{h\to 0} \frac{\left(500(x+h) - (x+h)^2 - 100\right) - \left(500x - x^2 - 100\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{500x} + 500h - \cancel{x^2} - 2xh - h^2 \cancel{-100} - \cancel{500x} + \cancel{x^2} \cancel{+100}}{h}$$

$$= \lim_{h\to 0} \frac{500h - 2xh - h^2}{h}$$

$$= \lim_{h\to 0} (500 + {-2x} - h)$$

$$= 500 - 2x$$

b.) Find and interpret $\overline{MP}(20)$

$\overline{MP}(20) = P'(20) = 460$

The profit from the sale of the 20th car is $460, ~~000~~.

c.) Find and interpret $\overline{MP}(300)$

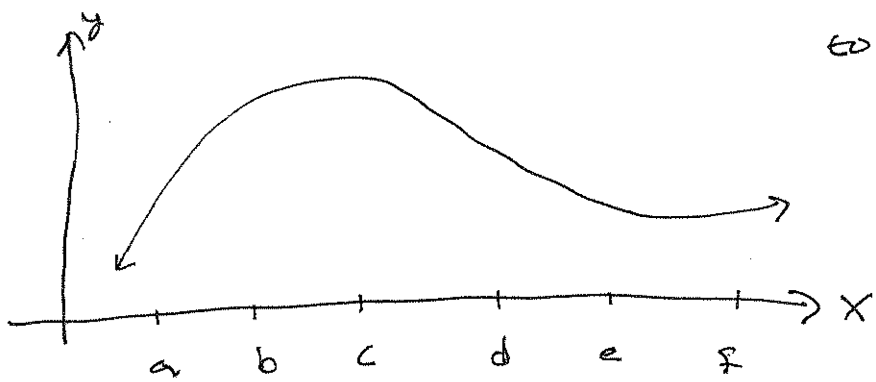
$\overline{MP}(300) = -100.$

You will lose $100, ~~000~~ on the sale of the 300th car.

Part 4: Graphical Derivative

Remember that the derivative at a point represents the slope of the tangent at that point.

Example 8: For the given graph, order $f'(a)$, $f'(b)$, ... from least to greatest



$f'(d) < f'(e) < f'(c) < f'(f) < f'(b) < f'(a)$

Example 9: For the given graph, what can be said about g' at a, b, ...?