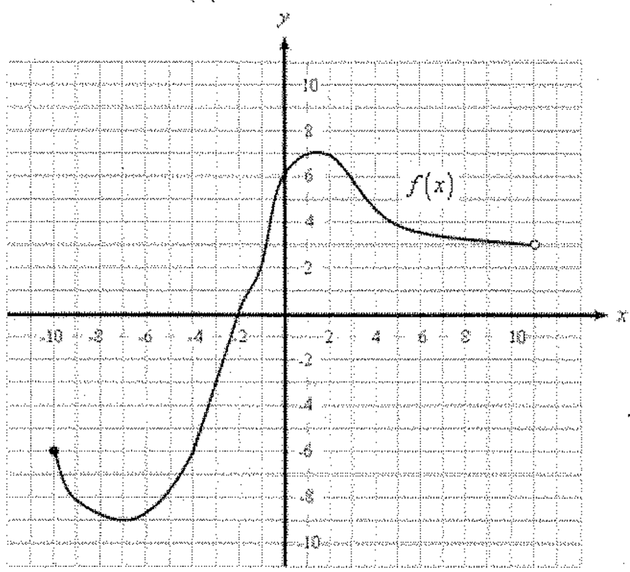
Limits

Section 9.1

Part 1: A Brief Review of Functions

Example 1: Functions and Graphs

Consider the complete graph of $f(x)$ that is given below.



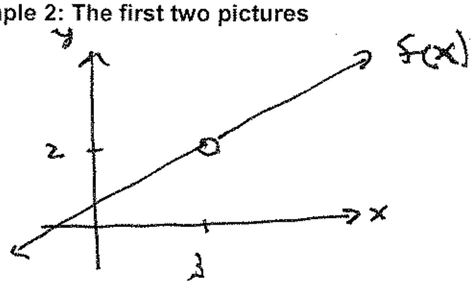
Use the graph to answer the following questions.

a.) $f(-10) =$ -6	b.) $f(-7) =$ -9
c.) $f(-2) =$ 0	d.) $f(0) =$ 6
e.) $f(3) =$ 6	f.) $f(11) =$ undefined.
g.) The domain of $f(x)$: $-10 \le x < 11$ or $[-10, 11)$	
h.) The range of $f(x)$: $-9 \le y \le 7$ or $[-9, 7]$	

Part 2: Graphical Limits

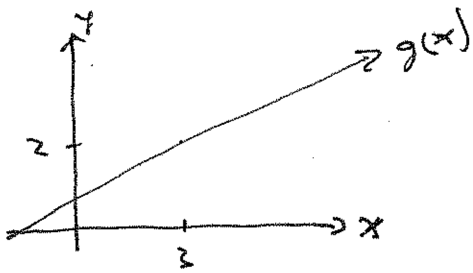
Example 2: The first two pictures

(a.)



the limit of f(x) as x approaches x=3 is y=2.

(b.)



the limit of g(x) as x approaches x=3 is y=2.

Definition: The Limit

Let $f(x)$ be a function defined on an open interval containing c, except possibly at $x = c$. Then

$\lim_{x \to c} f(x) = L$

if we can make values of $f(x)$ as close to L as we desire by choosing values of x sufficiently close to c.

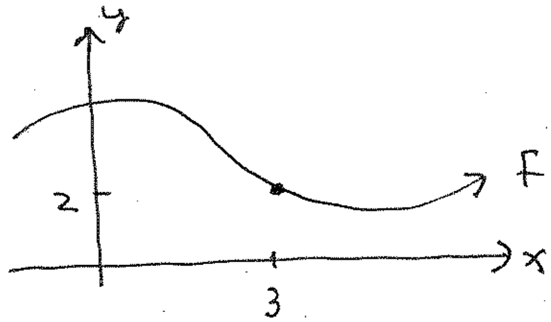
If the values of $f(x)$ do not approach a single finite L, the limit does not exist.

Notation: DNE means, "Does not exist."

Notation: We read $\lim_{x \to c} f(x) = L$ as, "The limit as x approaches c of $f(x)$ is L."

Example 3: Evaluating functions vs. evaluating limits

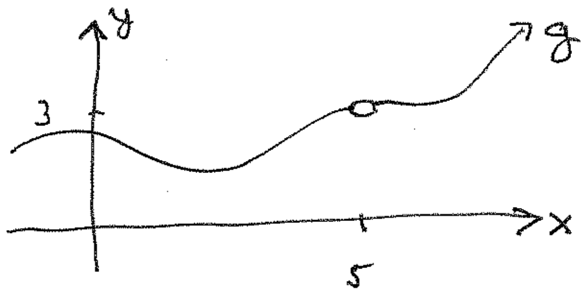
(a.)



(i) $f(3) = 2$

(ii) $\lim_{x \to 3} f(x) = 2$

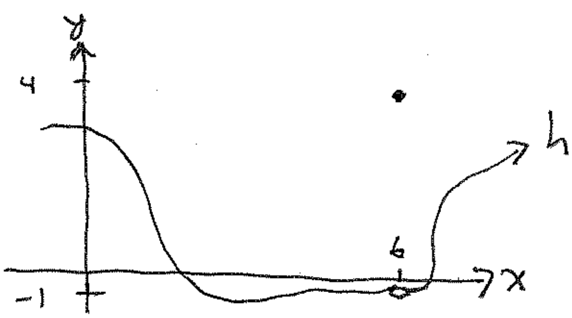
(b.)



(i) $g(5)$ is undefined.

(ii) $\lim_{x \to 5} g(x) = 3$

(c.)

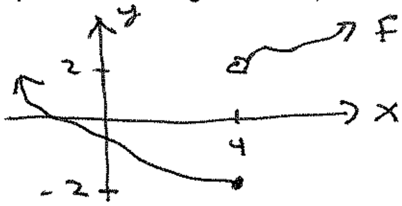


(i) $h(6) = 4$

(ii) $\lim_{x \to 6} h(x) = -1$

Example 4: Evaluating functions, left-hand and right-hand limits, limits, and limits at infinity.

(a.)



(i) $f(4) = -4$

(ii) $\lim_{x \to 4} f(x)$ DNE

(iii) $\lim_{x \to 4^+} f(x) = 2$

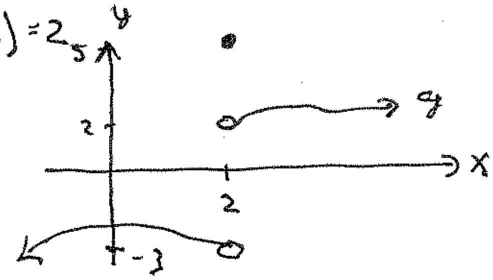
(b.)

(i) $g(2) = 5$

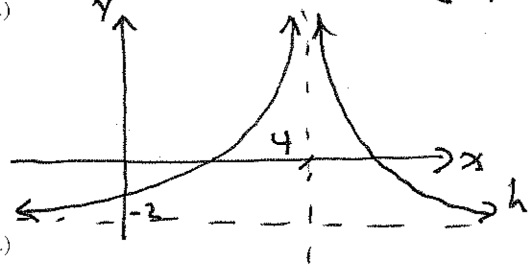
(ii) $\lim_{x \to 2} g(x)$ DNE

(iii) $\lim_{x \to 2^+} g(x) = 2$

(iv) $\lim_{x \to 2^-} g(x) = -3$



(c.)



(i) $h(4)$ undefined

(ii) $\lim_{x \to 4} h(x)$ DNE

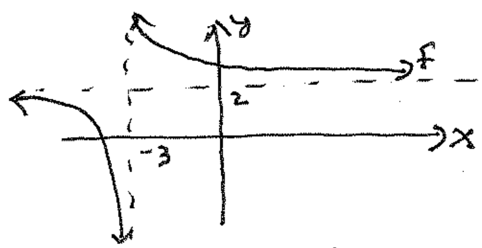
(iii) $\lim_{x \to 4^+} h(x) = \infty$

(iv) $\lim_{x \to 4^-} h(x) = \infty$

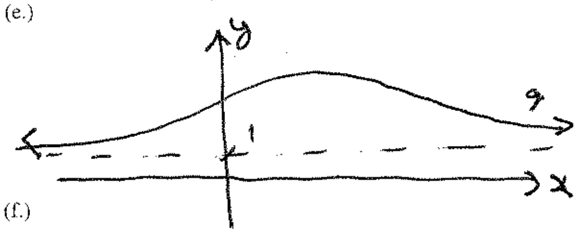
(d.)

(i) $\lim_{x \to -3^+} f(x) = \infty$

(ii) $\lim_{x \to -3^-} f(x) = -\infty$



(e.)



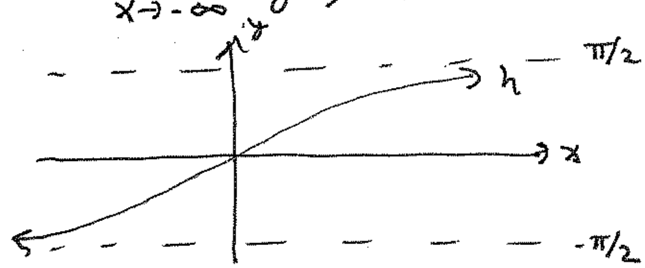
(i) $\lim_{x \to \infty} g(x) = 1$

(ii) $\lim_{x \to -\infty} g(x) = 1$

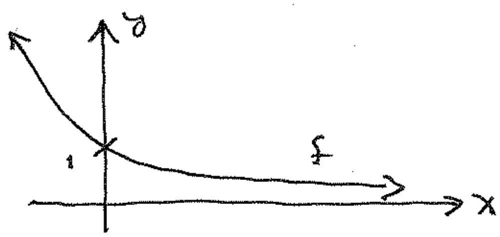
(f.)

(i) $\lim_{x \to \infty} h(x) = \pi/2$

(ii) $\lim_{x \to -\infty} h(x) = -\pi/2$



(g.)



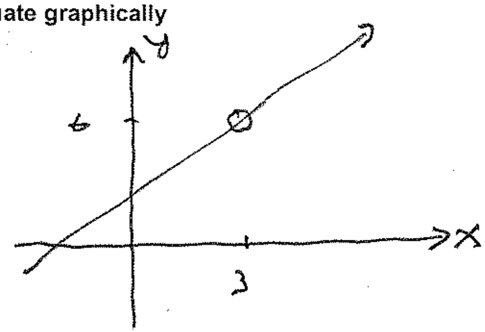
(i) $\lim_{x \to \infty} f(x) = 0$

(ii) $\lim_{x \to -\infty} f(x) = \infty$

Part 3: Limits Algebraically

Example 5: Evaluate graphically

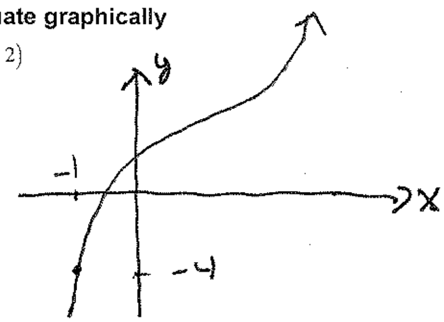
$\lim_{x\to 3} \frac{x^2-9}{x-3}$



$$\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$$

Example 6: Evaluate graphically

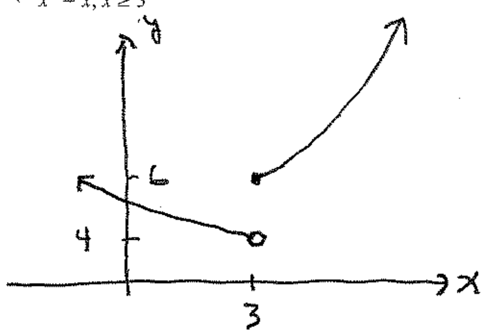
$\lim_{x\to -1} (4x^3 - 2x^2 + 2)$



$$\lim_{x\to -1} (4x^2 - 2x^2 + 2) = -1$$

Example 7: Evaluate graphically

$\lim_{x\to 3} f(x)$ where $f(x) = \begin{cases} 10 - 2x, & x < 3 \\ x^2 - x, & x \geq 3 \end{cases}$.



$\lim_{x\to 3} f(x)$ D.N.E.

Properties of Limits

If k is a constant, $\lim_{x\to c} f(x) = L$, and $\lim_{x\to c} g(x) = M$, then

I. $\lim_{x\to c} k = k$
II. $\lim_{x\to c} x = c$
III. $\lim_{x\to c} [f(x) \pm g(x)] = L \pm M$
IV. $\lim_{x\to c} [(f \cdot g)(x)] = L \cdot M$
V. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$
VI. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ provided that $L > 0$ when n is even.

limits of polynomials & rational fcts.

(I) If P a poly... $\lim_{x \to c} P(x) = P(c)$

(II) If R a rat fct s.t. $R(x) = \frac{N(x)}{D(x)}$ and $D(c) \neq 0$, then $\lim_{x \to c} R(x) = \lim_{x \to c} \frac{N(x)}{D(x)} = \frac{N(c)}{D(c)}$

Notation: "IFF" means "if and only if."

Definition: The Limit

$\lim_{x \to c} f(x) = L$ iff $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

That is, the limit from the right must equal the limit from the left in order for the limit to exist.

Example 6 revisited algebraically

$\lim_{x \to -1} (4x^3 - 2x^2 + 2)$

$$\underbrace{\lim_{x \to -1} (4x^3 - 2x^2 + 2)}_{\text{polynomial}} = \cancel{\lim_{x \to -1} 4x^3 - \lim_{x \to -1} 2x^2 + \underbrace{\lim_{x \to -1} 2}_{2}}$$

$$= -4 - 2 + 2$$

$$= -4$$

Example 7 revisited algebraically

$\lim_{x \to 3} f(x)$ where $f(x) = \begin{cases} 10 - 2x, & x < 3 \\ x^2 - x, & x \geq 3 \end{cases}$.

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (10 - 2x) = 4$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 - x) = 6$$

since $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$ we know $\lim_{x \to 3} f(x)$ DNE

Example 5 revisited algebraically

$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Can we find the limit algebraically?

The rule (II) for rat. fcts does not apply...

simplify.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x \to 3} (x+3)$$

$$= 6.$$

Evaluating limits at $x = c$ when the function is continious at $x = c$ is easy; simply evaluate the function at c.

Example 8

$\lim_{x\to 4} \frac{x^2-16}{x-4}$

$$= \lim_{x\to 4} (x+4)$$

$$= 8$$

Example 9

$\lim_{x\to 7} \frac{x^2-8x+7}{x^2-6x-7}$

$$= \lim_{x\to 7} \frac{\cancel{(x-7)}(x-1)}{\cancel{(x-7)}(x+1)}$$

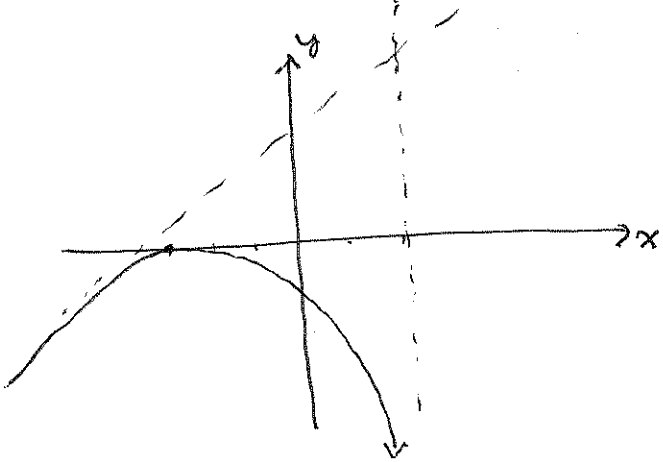
$$= \lim_{x\to 7} \frac{x-1}{x+1}$$

$$= \frac{6}{8}$$

Example 10

$\lim_{x\to 2} \frac{x^2+6x+9}{x-2}$

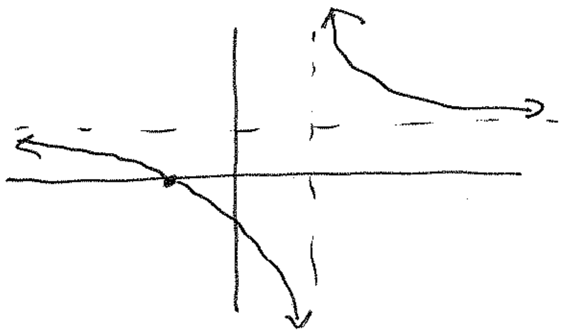
DNE (show graphically)



$$\begin{array}{r} x+4 \\ x-2\,\overline{)\,x^2+6x+9} \\ -(x^2-2x) \\ \hline 8x \end{array}$$

Example 11

$\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{(x-1)\cancel{(x-1)}}$ D.N.E.



Summary of Examples 8 - 11

Evaluating limits of rational functions where the denominator approaches zero.
a.) If the numerator does not approach zero, then the limit D.N.E. (does not exist).
b.) If the numerator approaches zero, simplify and then try again.

Example 12

$\lim_{x\to -1} f(x)$ where $f(x) = \begin{cases} x^2 + \frac{4}{x}, & x \le -1 \\ 3x^3 - x - 1, & x > -1 \end{cases}$ $= -3$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \left(x^2 + \frac{4}{x}\right) = -3$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1} (3x^3 - x - 1) = -3 + 1 - 1 = -3$$

Example 13

Suppose that the cost C of removing p percent of the pollution from an industrial plant is modeled by:

$$C(p) = \frac{730000}{100-p} - 7300$$

a.) Find and interpret $\lim_{p \to 80} C(p)$

\$29,200

The cost to remove 80% of the pollution is \$29,200.

b.) Find and interpret $\lim_{p \to 100^-} C(p)$ $= \infty$

There is an infinite cost if we wish to remove 100% of the pollution?

c.) Can all the pollution be removed?

Nope... Not unless we want to spend an infinite amount of cash.