Section 13.02

The Definite Integral and the Fundamental Theorem of Calculus

Part 1: The Definite Integral

To find the area under $f(x)$ (perhaps it makes the most sense to assume $f \geq 0$) on the interval $[a, b]$, we evaluate:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

We formalize this quantity through the definition of the definite integral given below.

Definition: The definite integral
If f is continuous on the interval $[a, b]$, then the area under f is given by:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\,\Delta x$ (the right end point of the i^{th} subinterval; each subinterval having equal width).

Example 1: Use the definition of the definite integral to write $\int_{-3}^{2} (4x - 7)\,dx$ as the limit of sums.

Now, the notation for the indefinite integral should lead to the obvious question: what is the relationship between the indefinite integral and the definite?

Part 2: The Fundamental Theorem of Calculus

Definition: The Fundamental Theorem of Calculus
Let f be continuous on the interval $[a, b]$. Then, the definite integral exists and:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. That is, $F' = f$.

Example 2: $\int_0^1 x\,dx$ (Note: This involves finding the area of the triangle that we worked three ways in the previous section).

Example 3: $\int_0^1 x^3\, dx$

Example 4: $\int_1^9 \sqrt{x}\, dx$

Example 5: $\int_0^5 4\sqrt[3]{x^2}\, dx$

Example 6: $\int_{2}^{4}(x^2+2)^3\,x\,dx$

Example 7: $\int_{-1}^{2}x\sqrt[3]{x^2-5}\,dx$

Example 8: Suppose that a vending machine service company models its income by assuming that money flows continuously into the machines with an annual rate of flow of $f(t) = 120\, e^{0.01\, t}$ where f gives the income in \$1,000/yr. Find the total income for the company over the first three years.