Section 12.03

Integrals of Exponentials and Logs

Part 1: The rules

Recall that $\frac{d}{dx} e^{u(x)} = u'(x) \cdot e^{u(x)}$. So:

Integral of an exponential

$$\int u'(x) \cdot e^{u(x)}\, dx = e^{u(x)} + C$$

And recall that $\frac{d}{dx} \ln(u(x)) = \frac{u'(x)}{u(x)}$. So:

Integral of a logarithm

$$\int \frac{u'(x)}{u(x)}\, dx = \ln|u(x)| + C$$

Question: Why is there an absolute value?

Part 2: Examples

Example 1: $\int 4 e^x \, dx$ $= 4\int e^x\, dx$

$= 4e^x + c$

Example 2: $\int 250\, e^{-\frac{1}{2}x}\, dx \;=\; 250 \int e^{-\frac{1}{2}x}\, dx$

Let $u = -\frac{1}{2}x$ $\quad = 250(-2)\int e^{-\frac{1}{2}x}\left(-\frac{1}{2}\right)dx$

$du = -\frac{1}{2}dx$

$= -500 \int e^{u}\, du$

$= -500\, e^{u} + c$

$= -500\, e^{-\frac{1}{2}x} + c$

Example 3: $\int \frac{x^3\, dx}{e^{4x^4}} \;=\; -\frac{1}{4}\int e^{-4x^4}\, 4x^3\, dx$

Let $u = -4x^4$ $\quad = -\frac{1}{4}\int e^{u}\, du$

$du = -4x^4 dx$

$= -\frac{1}{4}e^{u} + c$

$= -\frac{1}{4}e^{-4x^4} + c.$

Example 4: $\int \frac{3x^2}{x^3+4}\, dx \;=\; \int \frac{du}{u}$

Let $u = x^3+4$

$du = 3x^2 dx$

$= \ln|u| + c$

$= \ln|x^3+4| + c$

Example 5: $\int \frac{x^2\,dx}{x^3-9}$ $= \frac{1}{3}\int \frac{3x^2\,dx}{x^3-9}$

Let $u = x^3 - 9$ $= \frac{1}{3}\int \frac{du}{u}$

$du = 3x^2\,dx$

$= \frac{1}{3}\ln|u| + c$

$= \frac{1}{3}\ln|x^3 - 9| + c$

Example 6: $\int \frac{2x^3+x}{x^4+x^2}\,dx$ $= \frac{1}{2}\int \frac{2(2x^3+x)\,dx}{x^4+x^2}$

Let $u = x^4 + x^2$

$du = (4x^3 + 2x)\,dx$ $= \frac{1}{2}\int \frac{du}{u}$

$= 2(2x^3 + x)\,dx$

$= \frac{1}{2}\ln|u| + c$

$= \frac{1}{2}\ln|x^4 + x^2| + c$

Not every example will clearly lead to the natural logarithm. Some examples involve creativity. One of these creative approaches is to use polynomial long division.

Example 7: ~~$\int \frac{x^2+x+1}{x-1}\,dx$~~

$$\int \frac{2x^2+x-9}{x-2}\,dx = \int\left(2x+5+\frac{1}{x-2}\right)dx$$

$$= x^2+5x+\int\frac{dx}{x-2}$$

$$= x^2+5x+\ln|x-2|+c$$

$$\begin{array}{r} 2x+5 \\ x-2\,\overline{)\,2x^2+x-9} \\ -(2x^2-4x) \\ \hline 5x \\ -(5x-10) \\ \hline 1 \end{array}$$

Let $u=x-2$

$du=dx$

$\int\frac{dx}{x-2}=\int\frac{du}{u}$

$=\ln|u|+c$

Example 8: $\int \frac{x^4-2x^2+x}{x^2-2}\,dx$ $= \int\left(x^2+\frac{x}{x^2-2}\right)dx$

$$\begin{array}{r} x^2 \\ x^2-2\,\overline{)\,x^4+0x^3-2x^2+x+0} \\ (x^4-2x^2) \\ \hline 0+0+0+x \end{array}$$

$$= \frac{x^3}{3}+\frac{1}{2}\int\frac{2x}{x^2-2}\,dx$$

$$= \frac{x^3}{3}+\frac{1}{2}\int\frac{du}{u}$$

$$= \frac{x^3}{3}+\frac{1}{2}\ln|u|+c$$

$$= \frac{x^3}{3}+\frac{1}{2}\ln|x^2-2|+c$$

Let $u=x^2-$

$du=2x\,dx$

Part 3: Review problems

Example 9: $\int \left(3x^8 + \frac{4}{x^8} - \frac{5}{\sqrt[5]{x}}\right) dx = \int (3x^8 + 4x^{-8} - 5x^{-1/5})\, dx$

$$= \frac{3}{9}x^9 - \frac{4}{7}x^{-7} - \frac{5}{4/5}x^{4/5} + c$$

$$= \frac{1}{3}x^9 - \frac{4}{7}x^{-7} - \frac{25}{4}x^{4/5} + c$$

Example 10: $\int \frac{5x^3\, dx}{(x^4-8)^3} = \frac{5}{4}\int (x^4-8)^{-3}\, 4x^3 dx$

Let $u = x^4 - 8$
$du = 4x^3 dx$

$$= \frac{5}{4}\int u^{-3}\, du$$

$$= \frac{5}{4} \cdot \frac{u^{-2}}{-2} + c$$

$$= -\frac{5}{8}(x^4-8)^{-2} + c$$

Example 11: $\int \frac{x^2+1}{\sqrt{x^3+3x+10}}\, dx = \frac{1}{3}\int (x^3+3x+10)^{-1/2} \cdot 3(x^2+1)dx$

Let $u = x^3 + 3x + 10$
$du = (3x^2+3)dx$
$= 3(x^2+1)\, dx$

$$= \frac{1}{3}\int u^{-1/2}\, du.$$

$$= \frac{1}{3} \cdot \frac{u^{1/2}}{1/2} + c$$

$$= \frac{2}{3}(x^3+3x+10)^{1/2} + c$$

Example 12: $\int \left(x e^{3x^2} - \frac{5}{e^{\frac{x}{3}}}\right) dx = \frac{1}{6}\int 6x e^{3x^2} dx - (-3)\int \frac{5e^{-x/3}}{-3} dx$ ① ②

① Let $u = 3x^2$

$du = 6x\,dx$

$= \frac{1}{6}\int e^u\,du + 15\int e^v\,dv$

② Let $v = -\frac{x}{3}$

$dv = -\frac{1}{3}dx$

$= \frac{1}{6}e^u + 15e^v + C$

$= \frac{1}{6}e^{3x^2} + 15e^{-x/3} + C$

Example 13: $\int \frac{x+2}{x^2+4x-9} dx = \frac{1}{2}\int \frac{2(x+2)\,dx}{x^2+4x-9}$

Let $u = x^2+4x-9$

$du = (2x+4)dx$

$= 2(x+2)\,dx$

$= \frac{1}{2}\int \frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+4x-9| + C$

Example 13: $\cancel{\int \frac{x^4-2x^2+x}{x^2-2} dx}$ $\int \frac{4x^4+3x^3+4x^2+11x-24}{x^2+3} dx$

$$
\begin{array}{r}
4x^2+3x-8 \\
x^2+3 \overline{\smash{)}\,4x^4+3x^3+4x^2+11x-24} \\
-(4x^4 \qquad +12x^2) \\
\hline
3x^3-8x^2 \\
-(3x^3 \qquad +9x) \\
\hline
-8x^2+2x \\
-(-8x^2 \qquad -24) \\
\hline
2x+0
\end{array}
$$

$= \int\left(4x^2+3x-8+\frac{2x}{x^2+3}\right)dx$

$= \frac{4}{3}x^3 + \frac{3}{2}x^2 - 8x + \ln|x^2+3| + C$

Let $u = x^2+3$

$du = 2x\,dx$

$\int \frac{2x}{x^2+3}dx$

$= \int \frac{du}{u}$

$= \ln|u| + C$