Section 11.01

Derivatives of Logarithms

Part 1: Derivatives of Logs

If $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$.

Example 1: Find the derivatives of the following:

a.) $y = 4x^7 - 2\ln(x)$

$$y' = 28x^6 - \frac{2}{x}$$

b.) $f(x) = 3x^4 \ln(x)$

$$f'(x) = 12x^3 \ln x + 3x^4 \cdot \frac{1}{x}$$

c.) $g(x) = \frac{x^2}{\ln(x)}$

$$g'(x) = \frac{2x \ln x - \frac{1}{x} \cdot x^2}{[\ln(x)]^2}$$

If $f(x) = \ln(u(x))$, then $f'(x) = \frac{u'(x)}{u(x)}$ (the chain rule)

Example 2: Find the derivatives of the following:

a.) $y = \ln(x^4)$

$$y' = \frac{1}{x^4} \cdot 4x^3 = \frac{4}{x}$$

b.) $f(x) = \ln(x^3 - x + 7)$

$$f'(x) = \frac{1}{x^3 - x + 7} \cdot (3x^2 - 1)$$

c.) $z = \ln\left(\frac{2x^4}{(5x+7)^5}\right)$

$$\frac{dz}{dx} = \frac{1}{\frac{2x^4}{(5x+7)^5}} \cdot \frac{8x^3(5x+7)^5 - 5(5x+7)^4 \cdot 5 \cdot 2x^4}{[(5x+7)^5]^2}$$

Part 1: Derivatives with log rules

Logarithmic Rules: Let M, $N > 0$ and $p \in \mathbb{R}$ and $b > 0$ and $b \neq 1$.
1.) $\ln(e^x) = x$ (inverse function property)
2.) $e^{\ln(x)} = x,\ x > 0$ (inverse function property)
3.) $\ln(M \cdot N) = \ln(M) + \ln(N)$
4.) $\ln(\frac{M}{N}) = \ln(M) - \ln(N)$
5.) $\ln(M^p) = p \cdot \ln(M)$
6.) $\log_b(x) = \frac{\ln(x)}{\ln(b)}$ (change of base formula)

Example 2c revisited: Find the derivative of $z = \ln(\frac{2x^4}{(5x+7)^5})$ using log rules.

$$z = \ln(2x^4) - \ln[(5x+7)^5]$$
$$= \ln 2 + \ln x^4 - 5\ln(5x+7)$$
$$= \ln 2 + 4\ln x - 5\ln(5x+7)$$
$$\Rightarrow \frac{dz}{dx} = \frac{4}{x} - \frac{5}{5x+7} \cdot 5$$

Example 3: Find the derivatives of the following:

a.) $s = \ln(t^3(t^2 - 1))$

$= 3\ln t + \ln(t+1) + \ln(t-1)$

$\Rightarrow s' = \frac{3}{t} + \frac{1}{t+1} + \frac{1}{t-1}$

b.) $y = \ln\left(\sqrt[4]{\frac{3x+2}{x^2-5}}\right)$

$= \frac{1}{4}\ln\left(\frac{3x+2}{x^2-5}\right)$

$= \frac{1}{4}\left[\ln(3x+2) - \ln(x^2-5)\right]$

$\Rightarrow \frac{dy}{dx} = \frac{1}{4} \cdot \frac{1}{3x+2} \cdot 3 - \frac{1}{4} \cdot \frac{1}{x^2-5} \cdot 2x$

c.) $f(x) = \ln\left(x^2(x^4 - x + 1)^{17}\right)$

$= 2\ln x + 17\ln(x^4 - x + 1)$

$f'(x) = \frac{2}{x} + \frac{17}{x^4 - x + 1} \cdot (4x^3 - 1)$

Example 4: If the cost function for a product is $C(x) = 1500 + 200\ln(2x+1)$ where x is the number of units produced, then

a.) Find $\overline{MC}$

$$\overline{MC}(x) = \frac{200}{2x+1} \cdot 2$$

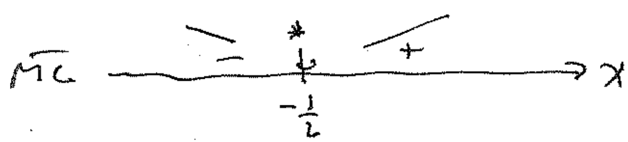
$$= \frac{400}{2x+1}$$

b.) Find and interpret $\overline{MC}(100)$

$$\overline{MC}(100) = \frac{400}{401} = 0.998.$$

The 100^{th} item costs about \$0.998 to produce.

c.) Does $C(x)$ always increase (does this result make sense)?



C decreases for neg. x's ... but these are outside the reasonable domain. So yes - it always increases which makes sense since products aren't free.

Example 5: Between 1976 and 1998, the percent of moms who returned to work within one year of having a baby can be represented by $w(y) = 1.11 + 16.94 \ln(y)$ where y is in years since 1970. What is the expected rate of change of w this year (and what does this mean)?

$$w'(y) = \frac{16.94}{y}$$

$$y'(40) = 0.4235$$

An additional 0.42 % will return to work compared to last year. (Approx).

Example 6: Find the following derivatives:

a.) $y = \log_4(x)$ $= \dfrac{\ln x}{\ln 4}$

$$\Rightarrow y' = \frac{1}{\ln 4} \cdot \frac{1}{x}$$

b.) $y = \log_6(x^4 - 4x^3 + 1)$

$$= \frac{\ln(x^4 - 4x^3 + 1)}{\ln 6}$$

$$\Rightarrow y' = \frac{1}{\ln 6} \cdot \frac{1}{x^4 - 4x^3 + 1} \cdot (4x^3 - 12x^2)$$