Section 9.6

The Chain Rule

Part 1: Composition of functions

Example 1: If $f(x) = 3x^4$ and $g(x) = 3 - 2x$, find $h(x) = f(g(x))$.

$h(x) = f(g(x)) = f(3-2x) = 3(3-2x)^4$

Example 2: For the following functions h, find f and g such that $h(x) = f(g(x))$.

a.) $h(x) = (2x^4 - 5)^{25}$

$g(x) = 2x^4 - 5$

$f(x) = x^{25}$

b.) $h(x) = (3 - 2x)^{10}$

$g(x) = 3 - 2x$

$f(x) = x^{10}$

c.) $h(x) = \frac{2}{3}(x^6 + 3x^2 - 11)^8$

$g(x) = x^6 + 3x^2 - 11$

$f(x) = \frac{2}{3}x^8$

Example 3: For the following functions *h*, find *f* and *g* such that $h(x) = f(g(x))$.

a.) $h(x) = \frac{1}{3(3x^2+3x+5)^{3/4}}$

$$g(x) = 3x^2 + 3x + 5$$

$$f(x) = \frac{1}{3x^{3/4}}$$

b.) $h(x) = \sqrt{x^2 + 3x}$

$$g(x) = x^2 + 3x$$

$$f(x) = \sqrt{x}$$

Part 2: The Chain Rule

Derivative Rule: The chain rule

If $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$

This can be memorized as, $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$.

Example 2 revisited: For each example, find $h'(x)$.

a.) $h(x) = (2x^4 - 5)^{25}$

$$h'(x) = 25(2x^4 - 5)^{24} \cdot 8x^3$$

b.) $h(x) = (3 - 2x)^{10}$

$$h'(x) = 10(3-2x)^9 \cdot (-2)$$

Example 2 revisited (continued from previous page)

c.) $h(x) = \frac{2}{3}(x^6 + 3x^2 - 11)^8$

$$h'(x) = \frac{16}{3}(x^6 + 3x^2 - 11)^7 (6x^5 + 6x)$$

Example 3 revisited: For each example, find $h'(x)$.

a.) $h(x) = \frac{1}{3(3x^2+3x+5)^{3/4}} = \frac{1}{3}(3x^2 + 3x + 5)^{-3/4}$

$$h'(x) = -\frac{1}{4}(3x^2 + 3x + 5)^{-7/4}(6x + 3)$$

b.) $h(x) = \sqrt{x^2 + 3x} = (x^2 + 3x)^{1/2}$

$$h'(x) = \frac{1}{2}(x^2 + 3x)^{-1/2} \cdot (2x + 3)$$

Example 4: Find the tangent line to $y = (x^2 + 1)^3$ at (2, 125).

$$y' = 3(x^2 + 1)^4 \cdot 2x$$

@ $x = 2$: $y' = 3 \cdot 5^4 \cdot 4 = 7500$

line: $y - 125 = 7500(x - 2)$

Example 5 (for you): Find the tangent line to $y = \left(\frac{1}{x^3 - x}\right)^3$ at $x = 2$.

$y = (x^3 - x)^{-3}$ ... pt $\left(2, \frac{1}{216}\right)$

$y' = -3(x^3 - x)^{-4}$

Example 6: Differentiate the following

a.) $y = \frac{5}{7}(2x^3 - x + 6)^{14}$

$y' = 10(2x^3 - x + 6)^{13}(6x^2 - 1)$

b.) $p = (q^3 + 1)^{-5}$

$p' = -5(q^3 + 1)^{-6} \cdot 3q^2$

c.) $f(x) = \frac{1}{(x^2+2)^3} = (x^2 + 2)^{-3}$

$f'(x) = -3(x^2 + 2)^{-4} \cdot 2x$

Example 7: Differentiate the following

a.) $g(x) = \frac{1}{(2x^3+3x+5)^{3/4}}$ $= (2x^3+3x+5)^{-3/4}$

$$g'(x) = -\frac{3}{4}(2x^3+3x+5)^{-7/4} \cdot (6x^2+3)$$

b.) $y = \frac{(3x+1)^5 - 3x}{7}$

$$y' = \frac{1}{7}\left[5(3x+1)^4 \cdot 3 - 3\right]$$

Example 8: $R(x) = 15(3x+1)^{-1} + 5x - 15$ gives the the dollars of revenue from the sale of x items. Find and interpret $\overline{MR}(4)$.

$$\overline{MR} = -15(3x+1)^{-2} \cdot 3 + 5$$

@ $x = 4$ $\overline{MR} = \frac{-45}{169} + 5$

$$= \frac{800}{169} \approx 4.73$$

The revenue from the sale of the 5th item (x goes from 4 to 5) is about \$4.73.

Quiz - Just for you

a.) Write down the product rule

$(u \cdot v)' = u'v + v'u$

b.) Write down the quotient rule

$\left(\frac{u}{v}\right)' = u'v - v'u$

c.) Write down the chain rule

If $h(x) = f(g(x))$, $h'(x) = f'(g(x)) \cdot g'(x)$

growth rate.

Example 9: The daily sales S attributed to an advertising campaign are given by the function $S(t) = 1 + \frac{3}{t+3} - \frac{18}{(t+3)^2}$ where t is the number of weeks the advertisement runs.

$S(t) = 1 + 3(t+3)^{-1} - 18(t+3)^{-2}$

a.) Find the ROC when $t = 8$.

$S'(t) = -3(t+3)^{-2} \cdot 1 + 36(t+3)^{-3} \cdot 1$

$S'(8) = \frac{-3}{11^2} + \frac{36}{11^3} \approx 0.00225$

b.) Find the ROC when $t = 10$.

$S'(10) = \frac{-3}{13^2} + \frac{36}{13^3} \approx -0.00137$

c.) Should the campaign continue after the tenth week?

Sales are decreasing... stop!