4:22

Test 3
Dusty Wilson
Math 220

$\bar{X} = 72.2\%$
med = 73.3%

Name: Key.

A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

David Hilbert
1862 – 1943 (Prussian mathematician)

No work = no credit

Warm-ups (1 pt each): $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \underline{-2}$ $\theta \cdot \theta^T = \underline{0}$ [1] $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \underline{-8}$

1.) (1 pt) Based upon your experience this quarter and the quote by Hilbert (above), how complete do you find linear algebra? Answer using complete English sentences.

L.A. is clear as mud & complete as my kids' jigsaw puzzles.

2.) (5 pts) An (n x n) matrix A has an eigenvalue $\lambda = 0$ with ~~algebraic~~ geometric multiplicity three. What is the rank of A?

nullity of $A = 3$

$\Rightarrow$ rank of $A = n - 3$.

2 pts - mercy
3 pts - if singular.

2 of the next 3 (10 pts).

3.) (5 pts) ~~Must~~ Give an example to show that the (n x n) matrix A ~~be~~ need not be invertible for a non-zero eigenvalue to exist. ~~If so, prove it. If not provide a counterexample.~~

No, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. A is singular, but $\lambda = 1$ is an eigenvalue.

2/5 if A^{-1} exists.

[1] Note to Tran and Hoa: The zero vector given lives in $\mathbb{R}^n$.

Give an example to show that not

4.) (5 pts) ~~Do~~ all (n x n) non-zero ~~matrix~~ matrices A have a non-zero eigenvalue? ~~If so, prove it. If not provide a counterexample~~.

~~Not~~ take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. $\lambda = 0$ is the only eigenvalue, but $A \neq \Theta$.

Find a nonzero matrix A where $\lambda = 0$ is the only eigenvalue.

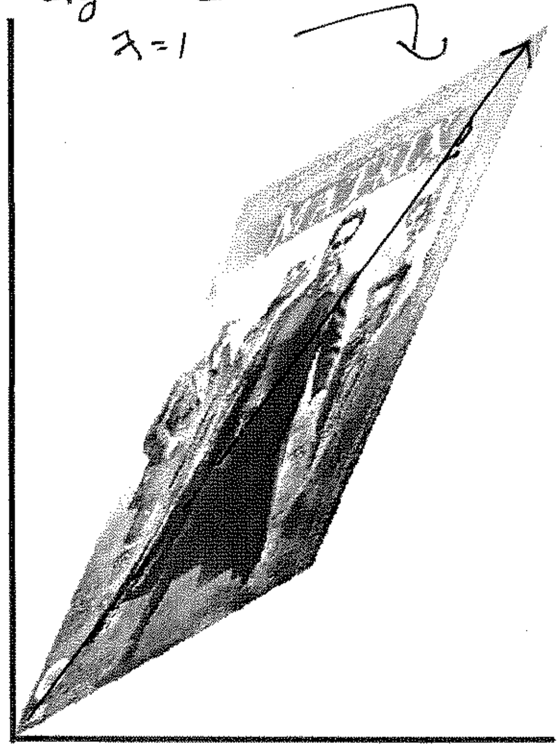
$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ works.

3/5 if $\lambda = 0$ is a eigenvalue.

5.) (5 pts) Suppose a transformation matrix is used to skew the image on the left into the image on the right. Clearly sketch the eigenvector(s) on both.

eigenvector

$\lambda = 1$



6.) ($\cancel{5}$ 20 pts) Find the eigenspace corresponding to one eigenvalue of the matrix

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$$

Solve $0 = |A - \lambda I|$

$$= \begin{vmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 5 \\ -2 & -1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -12 & 5 \\ 4 & -1-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -12 & -\lambda \\ 4 & -2 \end{vmatrix}$$ 5 pts.

$$= (3-\lambda)\left[\lambda + \lambda^2 + 10\right] + (12 + 12\lambda - 20) - (24 + 4\lambda)$$ 10 pts.

$$= 3\lambda + 3\lambda^2 + 30 - \lambda^2 - \lambda^3 - 10\lambda - 8 + 12\lambda - 24 - 4\lambda$$

$$= -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$$= -([\lambda^3 - 2\lambda^2] - [\lambda + -2])$$

$$= -(\lambda^2 - 1)(\lambda - 2)$$

$\Rightarrow \lambda = 2$ OR $\lambda = \pm 1$.

$\lambda = 2$: rref$([A - 2I | 0])$

$E_{\lambda=2} = \{x \mid x = a[1/2 \ -1/2 \ 1]^T, a \in \mathbb{R}\}$

$E_{\lambda=1} = \{x \mid x = a[3/7 \ -1/7 \ 1]^T, a \in \mathbb{R}\}$

2 pts $\rightarrow$ $E_{\lambda=-1} = \{x \mid x = a[1/2 \ 1 \ 1]^T, a \in \mathbb{R}\}$

$\boxed{\text{SPACE}}$

7.) (5 pts) Complete this $(2x2)$ matrix A (depending on a) so that its eigenvalues are $\lambda = \pm 1$.

$$C = \begin{bmatrix} a & 1 \\ c & d \end{bmatrix}$$

$$0 = |C - \lambda I| = \begin{vmatrix} a-\lambda & 1 \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - c$$

$$= ad - a\lambda - d\lambda + \lambda^2 - c$$

$$= \lambda^2 + (-a-d)\lambda + (ad - c)$$

$$= \lambda^2 - 1 \Rightarrow d = -a$$

$$ad - c = -1$$

$$\Rightarrow -a^2 - c = -1$$

$$\Rightarrow c = +1 - a^2$$

$$C = \begin{bmatrix} a & 1 \\ +1-a^2 & -a \end{bmatrix}$$

3/5 if for a specific a.

8.) (~~5~~ 10 pts) Find all eigenvalues and eigenvectors for the matrix $B = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix}$ given that $x = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ is an eigenvector. Do not calculate the characteristic polynomial.

$$Bx = \begin{bmatrix} 3 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 6i+4 \\ -2i+3 \end{bmatrix} \qquad \lambda = 3 - 2i$$

eigen vals. eigen vecs

$\lambda = 3 - 2i$ ~~$\lambda = 3 + 2i$~~ $x = a \begin{bmatrix} 2i \\ 1 \end{bmatrix}$

$\lambda = 3 + 2i$ $x = a \begin{bmatrix} -2i \\ 1 \end{bmatrix}$ $a \neq 0$ ← 2 pts.

9.) (~~5~~ 6 pts) Prove that if A is an $(n \times n)$ matrix with eigenvalue λ, then λ^k is an eigenvalue for A^k for $k = 2, 3, 4, \ldots$

$\Box$ proof by induction.

$k=2$: If $\exists\, x \neq \theta$ s.t. $Ax = \lambda x$

$\Rightarrow A(Ax) = A(\lambda x)$

$\Rightarrow A^2 x = \lambda (Ax) = \lambda^2 x.$

assume λ^k is an eigenvalue for A^k.

$k = n+1$: If $\exists\, x \neq \theta$ s.t. $A^k x = \lambda^k x$

$\Rightarrow A \cdot (A^k x) = A(\lambda^k x)$

$\Rightarrow A^{k+1} x = \lambda^k A x = \lambda^{k+1} x.$

Hence λ^{k+1} is an eigenval. of A^{k+1} ∎