$\bar{X} = 72.9\%$ 7:08

med = 77.9%

Test 2
Dusty Wilson
Math 220

Name: key

On earth there is nothing great but man;
in man there is nothing great but mind.

William Rowan Hamilton
1805 – 1865 (Irish mathematician)

No work = no credit

Warm-ups (1 pt each): $A \cdot \theta =$ θ $\quad \theta^T \cdot \theta =$ 0 $\quad \theta \cdot \theta^T =$ θ

1.) (1 pt) Do you agree with Hamilton that the greatest thing on earth is the human mind? Answer using complete English sentences.

No. Man is in the image of God. The shadow isn't as great as the original.

2.) (5 pts) What are the three methods we have discussed for finding the range of a matrix A?

(1) Transpose: non zero cols of $(\text{rref}(A^T))^T$

2 pts on explanation. (2) solve $[A|b]$ & determine conditions on b required for consistency.

(3) solve $Ax = \theta$. discard cols of A corresponding to arb. variables.

3.) (10 pts) Let W be the subset of $\mathbb{R}^3$ defined by $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_3 = 3x_2 - 7x_1 \right\}$. Prove that W is a subspace of $\mathbb{R}^3$.

(1) $\theta \in W$ ✓

(2) $u, v \in W \Rightarrow u = \begin{bmatrix} u_1 \\ u_2 \\ 3u_2 - 7u_1 \end{bmatrix}$ & $v = \begin{bmatrix} v_1 \\ v_2 \\ 3v_2 - 7v_1 \end{bmatrix}$

$$\Rightarrow u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ (3u_2 - 7u_1) + (3v_2 - 7v_1) \end{bmatrix}$$

7/10 if all components not shown.

$$= \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ 3(u_2 + v_2) - 7(u_1 + v_1) \end{bmatrix} \in W.$$

(3) u as above and $a \in \mathbb{R}$

$$au = a\begin{bmatrix} u_1 \\ u_2 \\ 3u_2 - 7u_1 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ a(3u_2 - 7u_1) \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ 3(au_2) - 7(au_1) \end{bmatrix} \in W$$

$\therefore au \in W \Rightarrow W$ is a subspace.

4.) (35 pts) Consider the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & 0 & 7 \end{bmatrix}$ and vector $v = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$.

a.) Find the null space and nullity of A. (Nullity = 1). Express your answer in set notation

solve $Ax = 0$

$\Rightarrow \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ $x_1 = -7x_3$
$x_2 = -3x_3$
x_3 arb.

$\Rightarrow N(A) = \{ x \mid x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -7x_3$ and $x_2 = -3x_3$, $x_3 \in \mathbb{R} \}$.

b.) Find the range and rank of A. (Rank = 2). ~~Express your answer in set notation~~.

$R(A) = Sp\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix} \right\}$

OR $Sp\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\} \leftarrow$ result using non zero col. of $(\text{rref}(A^T))^T$ & b s.t. $Ax = b$ is consistent.

c.) Is the vector v in the range of A? Justify your answer.

solve $Ax = v$. Inconsistent.

so $v \notin R(A)$.

d.) Find a basis for the range of A.

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}$

$$\begin{bmatrix}1\\0\\-3\end{bmatrix} \; \& \; \begin{bmatrix}3/5\\1\\1/5\end{bmatrix} \quad w/a = 3/5$$

e.) Find an orthogonal basis for the range of *A*.

$$u_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

$$u_2 = \overset{w_2}{\begin{bmatrix}-2\\-3\\0\end{bmatrix}} + a \overset{u_1}{\begin{bmatrix}1\\2\\1\end{bmatrix}} = \begin{bmatrix}-2\\-3\\0\end{bmatrix} + \frac{4}{3}\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-2/3\\-1/3\\4/3\end{bmatrix}$$

$$\Rightarrow a = -\frac{u_1^T w_2}{u_1^T u_1} = \frac{+8}{6} = +\frac{4}{3}$$

$$\begin{bmatrix}-2/3\\-1/3\\4/3\end{bmatrix}$$

f.) Find the best least-squares approximation w^* in the subspace $R(A)$ for the vector v. $= \begin{bmatrix}-1\\2\\6\end{bmatrix}$

$$w^* = a_1 u_1 + a_2 u_2$$

$$a_1 = \frac{u_1^T v}{u_1^T u_1} = \frac{9}{6} = \frac{3}{2}$$

$$a_2 = \frac{u_2^T v}{u_2^T u_2} = 8/\left(\tfrac{21}{9}\right) = \frac{72}{21}$$

$$\Rightarrow w^* = \frac{3}{2}\begin{bmatrix}1\\2\\1\end{bmatrix} + \frac{72}{21}\begin{bmatrix}-2/3\\-1/3\\4/3\end{bmatrix} = \cancel{\begin{bmatrix}-23/6\\1/3\\73/6\end{bmatrix}} = \begin{bmatrix}-11/14\\13/7\\85/14\end{bmatrix}$$

g.) Find the least-squares solution to the equation $Ax = v$.

solve $A^T A x = A^T v$.

$$\Rightarrow x^* = \begin{bmatrix}85/14 - 7x_3\\24/7 - 3x_3\\x_3\end{bmatrix}$$

or, solve $Ax = w^*$ for the same x^*

5.) (10 pts) Determine whether the function $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation. Justify your answer.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 x_3 \end{bmatrix}$$

No. if $a = 2$ & $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$T(ax) = \begin{bmatrix} 2+2 \\ 2 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \neq 2 \begin{bmatrix} 1+1 \\ 1 \cdot 1 \end{bmatrix} = aT(x).$$

6.) (5 pts) If the output vectors from Gram-Schmidt are $u_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $u_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$, describe all possible input vectors w_1 and w_2.

2pt if no search for inputs

$$u_2 = w_2 + a u_1$$

$$\Rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = w_2 + a \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\Rightarrow \quad w_2 = \begin{bmatrix} -\sin\theta - a\cos\theta \\ \cos\theta - a\sin\theta \end{bmatrix}.$$

AND $w_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

1 pt if p, not n.

7.) (8 pts) Prove the claim: Let $S = \{u_1, u_2, \ldots, u_n\}$ be a set of nonzero vectors in $\mathbb{R}^n$. If S is an orthogonal set of vectors, then S is a linearly independent set of vectors.

$\square$ proof.

suppose $\{u_1, \ldots, u_n\}$ be an orthogonal set of vectors in $\mathbb{R}^n$.

Let ~~$x \in \mathbb{R}^n$~~. $c_1, \ldots, c_n \in \mathbb{R}$ be s.t.

$$c_1 u_1 + \ldots + c_n u_n = \theta$$

$\Rightarrow$ $u_i^T(c_1 u_1 + \ldots + c_n u_n) = 0$ for $i = 1, \ldots, n$

$\Rightarrow$ $c_i u_i^T u_i = 0$ for $i = 1, \ldots, n$

$\Rightarrow$ $c_i = 0$ for $i = 1, \ldots, n$ since $u_i \neq \theta$.

$\Rightarrow$ S is L.I. $\blacksquare$