Similarity Transformations & Diagonalization

ex1: Suppose $A = \begin{bmatrix} 1 & 0 \\ 10 & 2 \end{bmatrix}$.

$\lambda_1 = 1$; $\lambda_2 = 2$.

$x_1 = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$; $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Form $S = \begin{bmatrix} -1 & 0 \\ 10 & 1 \end{bmatrix} \iff S^{-1} = -\begin{bmatrix} 1 & 0 \\ -10 & -1 \end{bmatrix}$

calculate $S^{-1}\underbrace{A\,S}$

$$\begin{bmatrix} 1 & 0 \\ 10 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 10 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 10 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ +10 & +1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

so $S^{-1}AS = D$ (a diagonal matrix).

ex2: calculate A^{10}.

$$(S^{-1}AS)^{10} = D^{10}$$

$$\Rightarrow S^{-1}A^{10}S = D^{10}$$

$$\Rightarrow A^{10} = S\,D^{10}\,S^{-1}$$

$$= \begin{bmatrix} -1 & 0 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 10 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 10 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 10240 & 1024 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 10230 & 1024 \end{bmatrix}$$

Dfn: The $(n \times n)$ matrices A & B are said to be similar if $\exists$ a nonsingular $(n \times n)$ matrix S s.t. $B = S^{-1} A S$.

This means B & $S^{-1}AS$ have the same eigenvals.

$$
\begin{aligned}
p(t) &= \det(\overbrace{S^{-1} A S}^{B} - tI) \\
&= \det(S^{-1}AS - tS^{-1}S) \\
&= \det[S^{-1}(A - tI)S] \\
&= \det(S^{-1}) \det(A - tI) \det(S) \\
&= \det(S^{-1}S) \det(A - tI) \\
&= \det(A - tI).
\end{aligned}
$$

Thus if A & B are similar $(n \times n)$ matrices, then A & B have the same eigenvalues. These eigenvals have the same alg. mult.

Note: Similar matrices have the same eigenvals, but not (necessarily) the same eigenvecs.

<u>Thm</u>: An $(n\times n)$ matrix A is diagonalizable iff A possesses a set of n L.I. eigenvecs.

□ <u>proof</u>. (constructive proof).

$(\Leftarrow)$: Suppose that $\{u_1,\dots,u_n\}$ is a set of L.I. eigenvecs for A:

$$A u_k = \lambda_k u_k, \quad k=1,\dots,n.$$

construct $S = [u_1,\dots,u_n]$

S is nonsingular $\Rightarrow$ S^{-1} exists.

$$S^{-1}S = [S^{-1}u_1,\dots,S^{-1}u_n] = [e_1,\dots,e_n] = I.$$

and $AS = [Au_1,\dots,Au_n] = [\lambda_1 e_1,\dots,\lambda_n e_n]$

$$\Rightarrow S^{-1}AS = [\lambda_1 S^{-1}u_1,\dots,\lambda_n S^{-1}u_n] = [\lambda_1 e_1,\dots,\lambda_n e_n]$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D \text{ (diagonal)}$$

$(\Rightarrow)$: Suppose $C^{-1}AC = D$, C nonsingular & D diagonal.

$C = [c_1,\dots,c_n]$ & $D = [d_1 e_1,\dots,d_n e_n]$

$C^{-1}AC = D \Rightarrow AC = CD$

and $AC = [Ac_1,\dots,Ac_n]$

$$CD = [d_1 Ce_1,\dots,d_n Ce_n]$$
$$= [d_1 c_1,\dots,d_n c_n]$$

so $AC = CD \Rightarrow Ac_i = d_i c_i$ for $i = 1, \ldots, n$

since C is non-singular $c_1, \ldots, c_n$ are LI and nonzero. These are n LI eigenvecs for A. ∎

<u>Thm</u>: Let A be an $(n \times n)$ matrix w/ n distinct eigenvals. Then A is diagonalizable.

<u>Dfn</u>: A real matrix Q is <u>orthogonal</u> if $Q^T Q = I$.

<u>Thm</u>: Let Q be an $(n \times n)$ orthogonal matrix.

(a) If $x \in \mathbb{R}^n$, then $\|Qx\| = \|x\|$ (length preserved)

(b) If $x, y \in \mathbb{R}^n$, then $(Qx)^T(Qy) = x^T y$ (angle preserved)

(c) $\mathrm{Det}(Q) = \pm 1$.

so far we have shown that <u>similar matrices</u> can be factored $(B = S^{-1} A S)$.

<u>Thm</u>: Let A be an $(n \times n)$ matrix where A has only real eigenvals. Then there is an $(n \times n)$ orthogonal matrix Q s.t. $Q^T A Q = T$ where T is an $(n \times n)$ upper-triangular matrix.

<u>Note</u>: This is a similarity transformation.

<u>Note</u>: this is called a schur decomposition.

compare

$B = S^{-1}AS$ B is not necessarily diagonal...

$T = Q^TAQ$ A has only real eigenvals

$D = S^{-1}AS$ A has n L.I. eigenvecs.

<u>Thm</u>: Let A be a real $(n \times n)$ matrix.

(a) If A is symmetric, then there is an orthogonal matrix Q s.t. $Q^TAQ = D$, where D is diagonal.

(b) If $Q^TAQ = D$, where Q is orthogonal & D is diagonal, then A is symmetric.

Q: Is this an iff statement? yes.

<u>Cor</u>: Let A be a real sym. matrix. It is possible to choose eigenvecs $u_1, \dots, u_n$ for A s.t. $\{u_1, \dots, u_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

NOTE: This may require Gram-Schmidt.

<u>Spectral Thm</u>: If A is an $(n \times n)$ symmetric matrix w/eigenvals $\lambda_1, \dots, \lambda_n$ & corresponding eigenvecs $u_1, \dots, u_n$. ~~If~~ We can know by the corollary that $u_1, \dots, u_n$ are orthonormal. Let $Q = [u_1, \dots, u_n]$.

$Q^TAQ = D = [\lambda_1 e_1, \dots, \lambda_n e_n]$

$\Rightarrow A = QDQ^T = \lambda_1 u_1 u_1^T + \dots + \lambda_n u_n u_n^T.$

<u>ex 3:</u> For the symmetric matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

find an orthogonal S s.t. $S^{-1}AS$ is diagonal.

The eigenvalues are $\lambda = 0$ (alg. mult. 2)

$\lambda = 3$

$$E_{\lambda=0} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$E_{\lambda=3} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

use Gram-Schmidt on $E_{\lambda=0}$

so $$E_{\lambda=0} = \text{span}\left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right)$$

$$E_{\lambda=3} = \text{span}\left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

AND if
$$S = \begin{array}{c} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \\ \begin{array}{ccc} u_1 & \quad u_2 & \quad u_3 \end{array} \end{array}$$

then
$$S^{-1}AS = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$
$$S^T A S = D$$

and $$A = SDS^T = u_1 \lambda_1 u_1^T + u_2 \lambda_2 u_2^T + u_3 \lambda_3 u_3^T$$