4.6: Complex Eigenvalues & Eigenvectors

Ex1: Find the eigenvalues & vectors of $A = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}$.

(A) $\det\left(\begin{bmatrix} 2-\lambda & 4 \\ -2 & -2-\lambda \end{bmatrix}\right) = (2-\lambda)(-2-\lambda) + 8$

$= -4 + \lambda^2 + 8$

$= \lambda^2 + 4.$

so $\lambda = \pm 2i$ are eigenvalues.

(B) Now, solve $(A - (2i)I)x = \theta$ for the eigenvector corresponding to $\lambda = 2i$.

$\left[\begin{array}{cc|c} 2-2i & 4 & 0 \\ -2 & -2-2i & 0 \end{array}\right]$ $\left(\frac{1}{4} + \frac{1}{4}i\right)R_1 \to R_1$

$\begin{bmatrix} 1 & 1+i & 0 \\ -2 & -2-2i & 0 \end{bmatrix}$ $R_2 + 2R_1 \to R_2$

$\begin{bmatrix} 1 & 1+i & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The non-trivial sol. is $x_1 = -(1+i)x_2$
x_2 arb.

so the eigenvector is $x = a\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$, $a \neq 0$.

<u>Thm</u>: Let A be a real $(n\times n)$ matrix w/ an eigenvalue λ & corresponding eigenvector x. Then $\bar{\lambda}$ is also an eigenvalue of A, and $\bar{x}$ is an eigenvector corresponding to $\bar{\lambda}$.

recall: if $z = a + bi$, $\bar{z} = a - bi$. The conjugate vector to $x = [x_1, \dots, x_n]^T$ is $\bar{x} = [\bar{x}_1, \dots, \bar{x}_n]^T$.

□ <u>proof of thm.</u>

It can be shown that $\overline{\lambda x} = \bar{\lambda}\bar{x}$.

And, it can be shown $\overline{Ax} = \bar{A}\bar{x} = A\bar{x}$ since A is real.

Now
$$\begin{aligned} A\bar{x} &= \overline{Ax} \\ &= \overline{\lambda x} \\ &= \bar{\lambda}\bar{x} \end{aligned}$$

Hence, $\bar{\lambda}$ is the eigenvalue corresponding to $\bar{x}$ ∎

<u>Thm</u>: If A is an $(n\times n)$ symmetric matrix, then all the eigenvalues of A are real.

□ <u>proof</u>.

Let A be an $(n\times n)$ real sym. matrix & suppose $Ax = \lambda x$, $x \neq 0$ & x possibly complex.

consider: $\bar{x}^T(Ax) = \bar{x}^T\lambda x = \lambda(\bar{x}^T x)$

$$\begin{aligned} \Rightarrow \lambda(\bar{x}^T x) = \bar{x}^T(Ax) &= (Ax)^T\bar{x} \\ &= x^T A^T \bar{x} \\ &= x^T A \bar{x} \quad (A \text{ sym}) \end{aligned}$$ ↙

↳ now since A is real, $A\bar{x} = \overline{\lambda x}$

$\Rightarrow \lambda \bar{x}^T x = x^T A \bar{x} = x^T \bar{\lambda} \bar{x} = \bar{\lambda} x^T \bar{x}$

$\Rightarrow \lambda (\bar{x}^T x) = \bar{\lambda}(x^T \bar{x})$

$x \neq \theta \Rightarrow \lambda = \bar{\lambda}$ and so λ is real ∎

ex1 cont: since $x = a \begin{bmatrix} -1-i \\ 1 \end{bmatrix}, a \neq 0$

is an eigenvector, $\bar{x} = a \begin{bmatrix} -1+i \\ 1 \end{bmatrix}, a \neq 0$

is also an eigenvec.

ex2: Find eigenvalues & vecs for $B = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$