4.5: Eigenvectors & Eigenspaces

Ex1: recall: If $B = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix}$, then

$p(t) = -t(t+1)$ & $\lambda = 0$
$\lambda = -1$ (mult 2)

are the eigenvalues. Find the eigenvectors.

$\lambda = 0$:

$\lambda = -1$: $\operatorname{rref}([B + 1I \mid \theta]) = \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow X = a \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$ corresponds to $\lambda = -1$ provided $a \neq 0$.

NOTE!

(1) ∞ many eigenvectors for any given λ.

(2) $\lambda = 0$ can be an eigenvalue.

(3) θ is never an eigenvector.

(4) The set of all eigenvectors corresponding to $\lambda = 0$ is the nullspace less θ.

(5) The eigenvectors of $A - \lambda I$ are the nonzero vecs in the null space of $A - \lambda I$.

Dfn Let A be an $(n \times n)$ matrix. If λ is an eigenvalue of A, then:

(a) The null space of $A - \lambda I$ is denoted by E_λ & is called the <u>eigenspace</u> of λ.

(b) The dimension of E_λ is called the <u>geometric multiplicity</u> of λ.

<u>ex1 rev:</u> The eigenspace of $\lambda = -1$ is

$$E_\lambda = \left\{ x \mid x = a\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}, \ a \in \mathbb{R} \right\}$$

The geometric mult. of $\lambda = -1$ is 1.

<u>ex2:</u> compare the alg. & geo. mult. for the eigenvalues of the following matrices.

(a) $C = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 4 & 2 \end{bmatrix}$

(b) $D = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

<u>Dfn</u>: Let A be an $(n \times n)$ matrix. If there is an eigenvalue λ of A s.t. the geo. mult. of λ is less than the alg. mult., then A is called a <u>defective</u> matrix.

<u>Q</u>: Which matrices we have seen are defective?

<u>Thm</u>: Let $u_1, \dots, u_k$ be eigenvectors of an $(n \times n)$ matrix A corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then $\{u_1, \dots, u_k\}$ is a L.I. set.

□ proof (by contradiction).

since $u_1 \neq \theta$, $\{u_1\}$ is a L.I. set.

suppose $\lambda_1, \dots, \lambda_k$ are distinct & $\{u_1, \dots, u_k\}$ are L.D.

$\Rightarrow \exists\, m$ on $2 \leq m \leq k$ s.t.

$S_1 = \{u_1, u_2, \dots, u_{m-1}\}$ is LI

$S_2 = \{u_1, u_2, \dots, u_m\}$ is L.D.

$\Rightarrow \exists\, c_1, \dots, c_m \in \mathbb{R}$ (not all zero) s.t.

** $$c_1 u_1 + c_2 u_2 + \dots + c_m u_m = \theta$$

we can hit both sides w/ A

$\Rightarrow c_1 A u_1 + \dots + c_m A u_m = \Theta$

$\Rightarrow c_1 \lambda_1 u_1 + \dots + c_m \lambda_m u_m = \Theta$ *

and w/ λ_m

$\Rightarrow c_1 \lambda_m u_1 + \dots + c_m \lambda_m u_m = \Theta$ *

we can take the difference of lines *

$$\Rightarrow c_1(\underbrace{\lambda_m - \lambda_1}_{\neq 0})u_1 + \dots + c_{m-1}(\underbrace{\lambda_m - \lambda_{m-1}}_{\neq 0})u_{m-1} = \Theta$$

$\Rightarrow \{u_1, \dots, u_{m-1}\}$ is L I so $c_1 = \dots = c_{m-1} = 0$.

$\therefore$ ~~$\{u_1, \dots, u_k\}$ is L.I.~~ Going back to ** on p3,
we have $c_1 u_1 + \dots + c_m u_m = \Theta + \dots + \Theta + c_m u_m = c_m u_m = \Theta$.
but $u_m \neq \Theta \Rightarrow c_m = 0$ & so S_2 is L.I. $\Rightarrow\Leftarrow$
$\therefore \{u_1, \dots, u_k\}$ are LI. ■

<u>cor:</u> Let A be an $(n \times n)$ matrix. If A has n distinct eigenvalues, then A has a set of n L.I eigenvectors.