4.1: Eigenvalue Problem for (2×2) matrices

The Eigenvalue Problem

For an $(n \times n)$ matrix A, find all scalars λ s.t. $Ax = \lambda x$ has a nonzero solution x.

such a scalar λ is an eigenvalue w/ corresponding eigenvector x.

What do eigenvectors/values do graphically.

How to find eigenvalues & vectors:

Solve $Ax = \lambda x$ for $x \neq \theta$

$$\Rightarrow \quad Ax - \lambda x = \theta$$

$$\Rightarrow \quad (A - \lambda I)x = \theta$$

If this is to have nonzero solutions, then $A - \lambda I$ must be ~~non~~ singular.

<u>step 1</u>: Find all scalars λ s.t. $A - \lambda I$ is ~~non~~ singular.

<u>step 2</u>: Given a scalar λ s.t. $A - \lambda I$ is singular, find all nonzero x's s.t. $(A - \lambda I)x = \theta$.

Eigenvalues for (2x2) matrices

$$\boxed{\begin{matrix} 4.1 \\ 2/2 \end{matrix}}$$

Recall that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ~~iff~~ is singular

iff $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0.$

<u>ex1:</u> Find the eigenvalues for $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

solve $0 = \det\left(\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda) + 1$

$$= 3 - \lambda - 3\lambda + \lambda^2 + 1$$

$$= \lambda^2 - 4\lambda + 4$$

$$= (\lambda - 2)^2.$$

so $\lambda = 2$ is the only eigenvalue. (mult. 2).

<u>ex2:</u> Find the eigenvectors

solve $(A - 2I)x = \theta$ for nonzero x.

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} x = \theta$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 = -x_2$$

the eigenvector(s) corresponding to $\lambda = 2$

is $a\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $a \neq 0$.

History of Eigenvalues and Eigenvectors

Eigenvalues are often introduced in the context of linear algebra or matrix theory. Historically, however, they arose in the study of quadratic forms and differential equations.

Euler studied the rotational motion of a rigid body and discovered the importance of the principal axes. Lagrange realized that the principal axes are the eigenvectors of the inertia matrix.[1] In the early 19th century, Cauchy saw how their work could be used to classify the quadric surfaces, and generalized it to arbitrary dimensions.[2] Cauchy also coined the term *racine caractéristique* (characteristic root) for what is now called *eigenvalue*; his term survives in *characteristic equation*.[3]

Fourier used the work of Laplace and Lagrange to solve the heat equation by separation of variables in his famous 1822 book *Théorie analytique de la chaleur*.[4] Sturm developed Fourier's ideas further and brought them to the attention of Cauchy, who combined them with his own ideas and arrived at the fact that real symmetric matrices have real eigenvalues.[2] This was extended by Hermite in 1855 to what are now called Hermitian matrices.[3] Around the same time, Brioschi proved that the eigenvalues of orthogonal matrices lie on the unit circle,[2] and Clebsch found the corresponding result for skew-symmetric matrices.[3] Finally, Weierstrass clarified an important aspect in the stability theory started by Laplace by realizing that defective matrices can cause instability.[2]

In the meantime, Liouville studied eigenvalue problems similar to those of Sturm; the discipline that grew out of their work is now called *Sturm-Liouville theory*.[5] Schwarz studied the first eigenvalue of Laplace's equation on general domains towards the end of the 19th century, while Poincaré studied Poisson's equation a few years later.[6]

At the start of the 20th century, Hilbert studied the eigenvalues of integral operators by viewing the operators as infinite matrices.[7] He was the first to use the German word *eigen* to denote eigenvalues and eigenvectors in 1904, though he may have been following a related usage by Helmholtz. "Eigen" can be translated as "own", "peculiar to", "characteristic", or "individual"—emphasizing how important eigenvalues are to defining the unique nature of a specific transformation. For some time, the standard term in English was "proper value", but the more distinctive term "eigenvalue" is standard today.[8]

The first numerical algorithm for computing eigenvalues and eigenvectors appeared in 1929, when Von Mises published the power method. One of the most popular methods today, the QR algorithm, was proposed independently by John G.F. Francis[9] and Vera Kublanovskaya[10] in 1961.[11]