Least squares.

3.8
1/3

consider $Ax = b$ where A is $(m \times n)$.
If $x \in \mathbb{R}^n$ then $r = Ax - b$ is the residual vector & x^* that yields the smallest possible r is the least squares solution to $Ax = b$. That is

$$\|Ax^* - b\| \leq \|Ax - b\| \text{ for all } x \in \mathbb{R}^n$$

Tomorrow we will derive the formula... today we will use it... The least squares solution to $Ax = b$ is $A^T A x = A^T b$.

<u>ex1:</u> $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -7 \\ 1 & 3 & 5 \end{bmatrix}$ & $b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

<u>ex2:</u> linear fit to:

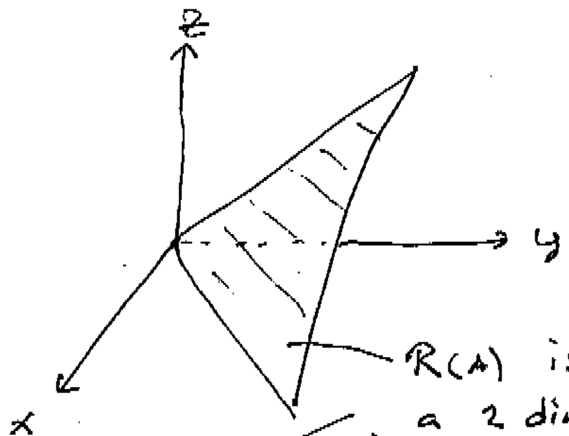
t	-2	0	1	2
y	2	1	0	0

<u>ex3:</u> quadratic fit to:

t	0	1	2	3
y	0	0	1	2

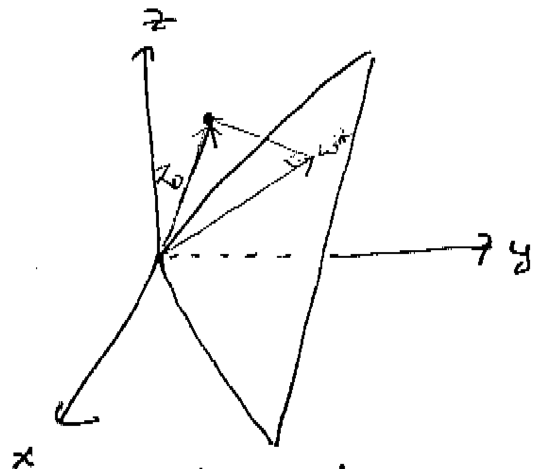
3.8
2/3

To understand the origin of the formula $A^T A x = A^T b$ for finding least squares sol. consider an inconsistent ~~sys~~ (3×2) sys.



R(A) is a 2 dim subspace in $\mathbb{R}^3$.

$Ax = w \leftarrow w \in R(A)$



w^* is the vector/pt in R(A) closest to $\vec{b}$.

$\Rightarrow \quad \|w^* - b\| \leq \|w - b\| \quad \forall\, w \in R(A).$

we can see that $(w^* - b) \perp$ to every vector in R(A). in particular, it is $\perp$ to the cols of A which are a spanning set for R(A).

$$\Rightarrow \quad \begin{aligned} A_1^T (w^* - b) &= 0 \\ A_2^T (w^* - b) &= 0 \end{aligned}$$

$\Rightarrow \quad A^T (w^* - b) = \theta$

But $w^* \in R(A) \Rightarrow \exists\, x^* \in \mathbb{R}^2$ s.t. $Ax^* = w^*$.

$\Rightarrow \quad A^T (Ax^* - b) = 0$

$\Rightarrow \quad A^T A x^* = A^T b.$

<u>Thm</u>: Consider the $(m \times n)$ sys. $Ax = b$.

(1) The associated sys. $A^T A x = A^T b$ is always consistent.

(2) The least squares sol. of $Ax = b$ are precisely the sol. of $A^T A x = A^T b$.

(3) The least-squares sol. is unique iff A has rank n.

<u>Dfn</u>. If the rank of A is less than n, then we say A is rank difficient.

Concern w/ round-off.

<u>ex4</u>: $A = \begin{bmatrix} 120 & 148 & 175 & 204 & 232 & 260 & 288 & 316 & 345 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

$b = [3, 4, 5, 5.5, 6, 7.5, 8.8, 10, 11.1, 12]^T$

check out $A^T A$ & $A^T b$....