Linear Transformations

3.7
1/4

Dfn: Let V & W be subspaces of $\mathbb{R}^n$ & $\mathbb{R}^m$ respectively. Let T be a fct from V to W; $T: V \mapsto W$. We say T is a <u>linear transformation</u> if $\forall\, u, v \in V$ & $a \in \mathbb{R}$

(1) $T(u+v) = T(u) + T(v)$

(2) $T(a u) = a\,T(u)$.

Ex1: determine if $F: \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = |x_1| + |x_2|$ is a linear transformation.

Ex2: determine if $F: \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined by $F\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_2 \end{bmatrix}$ is a linear transformation

Ex3: determine if $F: \mathbb{R} \mapsto \mathbb{R}$ defined by $F([x_1]) = [2x_1 + 3]$ is a linear transformation.

3.7
2/4

<u>Ex 4:</u> Let W be a subspace of $\mathbb{R}^n$ w/ $\dim(W) = p$ w/ orthonormal basis $S = \{w_1, \dots, w_p\}$. Define $T: \mathbb{R}^n \mapsto W$ by $T(v) = (v^T w_1) w_1 + \dots + (v^T w_p) w_p$.

Prove T is a linear transformation.

(1) $T(u + v) = T(u) + T(v)$ is in the book.

(2) Let $u \in W$ & $c \in \mathbb{R}$ be given. Now

$$
\begin{aligned}
T(cu) &= ((cu)^T w_1) w_1 + \dots + ((cu)^T w_p) w_p \\
&= c(u^T w_1) w_1 + \dots + c(u^T w_p) w_p \\
&= c\left((u^T w_1) w_1 + \dots + (u^T w_p) w_p \right) \\
&= c\, T(u).
\end{aligned}
$$

This is called the orthogonal projection of v onto W.

There is a simple illustration in the book where $T: \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$

Two other simple linear transformations

(1) $T(v) = a v$ (a dilation or contraction).
when $a = 1 \Rightarrow$ identity transformation.

(2) $T(v) = \theta$ (the zero transformation)

3.7 3/4

What is the relationship between a linear transformation & a matrix? Let's look @ a few examples...

ex5: Suppose W is the subspace of $\mathbb{R}^3$ defined by $W = \{ x \mid x = \begin{bmatrix} x_1 \\ x_3 - 4x_1 \\ x_3 \end{bmatrix} \text{ for } x_1, x_3 \in \mathbb{R} \}$

$\{w_1, w_2\} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

Suppose we know $T: W \mapsto \mathbb{R}^2$ is s.t.

$$T(w_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \& \quad T(w_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find $T\left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right)$ ...

Notice that no matrices were used...

The action of a linear transformation is completely known once we know the action of the transformation of a basis.

ex6: Let $T: \mathbb{R}^4 \mapsto \mathbb{R}^2$ s.t.

$$T(e_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad T(e_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \quad T(e_3) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}; \quad T(e_4) = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

For an arb. vec. $x = [x_1 \ x_2 \ x_3 \ x_4]^T \in \mathbb{R}^4$.

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$$

$$\Rightarrow T(x) = x_1 T(e_1) + \dots + x_4 T(e_4)$$

3.7
4/6

$$= x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

$\Rightarrow$ Then $T(x) = Ax$ for $x \in \mathbb{R}^4$
when $A = [T(e_1), \dots, T(e_4)]$.

In General, let $T: \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation & let $e_1, \dots, e_n$ be the unit vecs in $\mathbb{R}^n$. If A is the $(m \times n)$ matrix defined by

$$A = [T(e_1), \dots, T(e_n)]$$

then $T(x) = Ax$.

<u>Dfn</u>: Let V & W be subspaces & $T: V \to W$ be a lin. trans.

null space: $\mathcal{N}(T) = \{v \mid v \in V \text{ \& } T(v) = \theta\}$

range $\mathcal{R}(T) = \{w \mid w \in W \text{ \& } w = T(v) \text{ for some } v \in V\}$.

What is the difference between the definitions & those for matrices?

explain (ex9) on p234 in the text...

<u>ex6 rev</u>: Find the nullity & rank of T...

3.7
5/6

Derive the formula for T if:

$$T\left(\overset{u_1}{\begin{bmatrix}1\\0\\1\end{bmatrix}}\right) = \begin{bmatrix}0\\-1\\1\end{bmatrix};\quad T\left(\overset{u_2}{\begin{bmatrix}0\\-1\\1\end{bmatrix}}\right) = \begin{bmatrix}2\\1\\0\end{bmatrix};\quad T\left(\overset{u_3}{\begin{bmatrix}1\\-1\\0\end{bmatrix}}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

we want $T\left(\overset{x}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}\right) = x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$

so we need $T(e_1)$, $T(e_2)$, $T(e_3)$, which means we need e_1, e_2, e_3 in terms of u_1, u_2, u_3.

<u>method 1</u>: by observation.

$$\frac{1}{2}(u_1 + u_2 - u_3) = e_3$$

$$\Rightarrow T(e_3) = \frac{1}{2}\left(T(u_1) + T(u_2) - T(u_3)\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}0\\-1\\1\end{bmatrix} + \begin{bmatrix}2\\1\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

<u>method 2</u>: using row reduction.

The same.

$$\operatorname{rref}\left(\begin{array}{c} \begin{array}{cccccc} & & & e_1 & e_2 & e_3 \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}\right)$$

$$\frac{1}{2}u_1 + \frac{1}{2}u_2 - \frac{1}{2}u_3 = e_3$$

$$= \left[\begin{array}{c|ccc} & 1/2 & 1/2 & 1/2 \\ I & -1/2 & -1/2 & 1/2 \\ & 1/2 & -1/2 & -1/2 \end{array}\right]$$

$$\Rightarrow T(e_1) = \frac{1}{2}\left(\begin{bmatrix}0\\-1\\1\end{bmatrix} - \begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

$\boxed{\begin{matrix} 3.7 \\ 6/6 \end{matrix}}$

and $T(e_2) = \frac{1}{2}\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

Hence $T(x) = x_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -x_1 - x_2 + x_3 \\ -x_1 - x_2 \\ x_1 \end{bmatrix}$$

or $T(x) = Ax$ where $A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.