3.6: Orthogonal Bases for Subspaces

vectors u & v are orthogonal if $u^T v = 0$.

A set of vectors $S = \{u_1, \dots, u_p\}$ is orthogonal if each distinct pair of vectors in S is orthogonal.

ex1: verify $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \right\}$ is an orthogonal set of vectors.

Thm: Let $S = \{u_1, \dots, u_p\}$ be a set of non zero vectors in $\mathbb{R}^n$. If S is an orthogonal set of vectors, then S is a L.I. set of vectors.

□ proof.

Let $c_1, \dots, c_p \in \mathbb{R}$ be s.t.

$$c_1 u_1 + \dots + c_p u_p = \Theta.$$

$\Rightarrow u_1^T(c_1 u_1 + \dots + c_p u_p) = u_1^T \Theta$

$\Rightarrow c_1 u_1^T u_1 + 0 + \dots + 0 = 0$

$\Rightarrow c_1 = 0$ since $u_1 \neq \Theta$.

the same is true for $c_2, \dots, c_p \Rightarrow S$ is LI ■

ex2: normalize S from (ex1).

Dfn: An orthonormal basis is an orthogonal basis where each vector has unit length.

3.6 2/3

Cor: Let W be a subspace of $\mathbb{R}^n$ where $\dim(W) = p$. If S is an orthogonal set of p nonzero vectors & is also a subset of W, then S is an orthogonal basis for W.

coordinates & orthogonal bases.

If $B = \{v_1, \dots, v_p\}$ is a basis for W and $u \in W \Rightarrow \exists\, a_1, \dots, a_p \in \mathbb{R}$ s.t. $u = a_1 v_1 + \dots + a_p v_p$. We call $a_1, \dots, a_p$ the coords of u WRT B.

ex 3: Find the coords of $\vec{u} = [2 \quad 2 \quad 0 \quad 2]^T$

WRT
$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$
Ans $(2, 1, -1)$

ex 4: using the vecs from (ex 1) find the coords of $\vec{u} = [1 \quad 2 \quad 3]^T$.

$$\vec{u} = a_1 w_1 + a_2 w_2 + a_3 w_3$$
$$w_1^T \vec{u} = a_1 w_1^T w_1 \Rightarrow a_1 = \frac{w_1^T u}{w_1^T w_1}$$
$$a_2 = \frac{w_2^T u}{w_2^T w_2}$$
$$a_3 = \frac{w_3^T u}{w_3^T w_3}$$

$\boxed{\begin{matrix} 3.6 \\ 3/3 \end{matrix}}$

To find/create an orthogonal basis we will start w/ basis & use the Gram-Schmidt process.

$$\{w_1, \ldots, w_p\}$$

$$\downarrow$$

$$u_1 = w_1$$

$$u_2 = w_2 + a u_1$$

$$u_3 = w_3 + b u_1 + c u_2$$

$$\vdots$$

key step:

$$u_2 = w_2 + a u_1$$

$$\underbrace{u_1^T u_2} = u_1^T w_2 + a\, u_1^T u_1$$

$$0 = u_1^T w_2 + a\, u_1^T u_1$$

$$\Rightarrow a = -\frac{u_1^T w_2}{u_1^T u_1}$$

<u>ex 5</u>: Find an orthogonal basis for $sp(B)$ (as B is given in ex3).