Bases for Subspaces

Two bases you know.

(1) any pt (x,y,z) can be represented/reached by $xe_1 + ye_2 + ze_3$.

(2) $\vec{T}, \vec{N}, \& \vec{B}$ ~~#~~ along a space curve.

You know the concept of the span. Now ~~is~~ a spanning set for a subspace W is any set of vectors $S = \{w_1, \dots, w_m\}$ s.t. $W = Sp\{w_1, \dots, w_m\}$.

ex 1: spanning sets for $\mathbb{R}^2$.

(a) $\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\}$, (b) $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\4\end{bmatrix} \right\}$.

show (a) is a spanning set for $\mathbb{R}^2$.

$$\begin{bmatrix}1 & 1 & 1 & a\\ 1 & 0 & -1 & b\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 1 & a\\ 0 & 1 & 2 & a-b\end{bmatrix}$$

$$\begin{bmatrix}a\\b\end{bmatrix} = x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\0\end{bmatrix} + x_3\begin{bmatrix}1\\-1\end{bmatrix} \quad \Rightarrow \begin{bmatrix}1 & 0 & -1 & b\\ 0 & 1 & 2 & a-b\end{bmatrix}$$

$x_1 = b + x_3$

$x_2 = (a-b) - 2x_3$

x_3 is arb.

ex 2: Find a spanning set for $N(A)$ if
$$A = \begin{bmatrix} -3 & -5 & 36 \\ -1 & 0 & 7 \\ 1 & 1 & -10 \end{bmatrix}$$ (as on quiz 2).

$$N(A) = \{ x \mid x_1 = 7x_3 \ \& \ x_2 = 3x_3 \}.$$
$$= sp\left\{ \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Goal: We want to be efficient, so we want to include the fewest possible vectors in our spanning set. We call this a minimal spanning set.

The vectors in a minimal spanning set must be L.I. If $S = \{w_1, \dots, w_m\}$ is a L.D. spanning set for W then at least 1 vector in W can be discarded to produce a smaller spanning set. Why? If S is LI ... no vectors can be removed.

Dfn. Let W be a nonzero subspace of $\mathbb{R}^n$. A basis for W is a L.I. spanning set for W.

Given a basis $B = \{v_1, \dots, v_p\}$ for W, a subspace of $\mathbb{R}^n$, $x \in W$ has a unique rep.
$x = a_1 v_1 + \dots a_p v_p$. If not, $\exists\, b_1, \dots, b_p$ s.t.
$x = b_1 v_1 + \dots + b_p v_p \Rightarrow x - x = \theta = (a_1 - b_1)v_1 + \dots + (a_p - b_p)v_p$
But this is the homogeneous eqt. since B is LI it has only the trivial sol $\&$ $a_1 = b_1 \dots a_p = b_p \Rightarrow\Leftarrow$

ex3: Find the range & a basis for the range of

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 0 \\ -1 & 2 & 2 \end{bmatrix}$$

ex4: Let W be the subset of $\mathbb{R}^4$ spanned by

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}; \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}; \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \quad v_4 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \end{bmatrix}; \quad v_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Find a basis for W.

A basis must be L.I. so we can solve

$x_1 v_1 + x_2 v_2 + \dots + x_5 v_5 = \theta$ & find the v_i that can be eliminated.

Thm: If the nonzero matrix A is row equiv. to the matrix B in echelon form, then the nonzero rows of B form a basis for the row space of A.

ex4 new:
$$V = \begin{bmatrix} 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 2 & 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 1 \\ 2 & 1 & 1 & -1 \\ 0 & 2 & 2 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

row reduce & transpose.

The nonzero cols form a basis.