Four Special Subspaces

(1) SPAN

(2) NULL SPACE OR KERNEL

(3) RANGE OR COLUMN SPACE

(4) ROW SPACE.

Two ideas (one old & one new).

(1) LINEAR TRANSFORMATION. IF $A_{n\times m}$ then $Ax = b$

$A : \mathbb{R}^m \longmapsto \mathbb{R}^n$

(2) LINEAR COMBINATION.

$\vec{X}$ is a lin. comb of $v_1, \ldots, v_n$ if $\exists\ a_1, \ldots, a_n$ s.t.

$\vec{x} = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$.

(1) <u>The Span</u>

If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ then the span of the vectors is the supspace $sp\{\vec{v}_1, \ldots, \vec{v}_m\}$ made up of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$

confirm it is a subspace.

(1) $0 \cdot \vec{v}_1 + \ldots + 0 \cdot \vec{v}_m = \theta$ is in the span $\Rightarrow \theta$ is in the space.

(2) Take $x, y \in$ span. $\Rightarrow x = a_1 v_1 + a_2 v_2 + \ldots + a_m v_m$ and $y = b_1 \vec{v}_1 + \ldots + b_m \vec{v}_m$ for some a_i & b_i.

$\Rightarrow x + y = (a_1 + b_1) v_1 + \ldots + (a_m + b_m) v_m \in$ span since it is a lin. comb. of $v_1, \ldots, v_m$.

(3) Take $x \in$ span & $a \in \mathbb{R}$

$\Rightarrow ax = a(a_1 v_1 + \ldots + a_m v_m) = (a a_1) v_1 + \ldots + (a a_m) v_m \in$ span.

Hence $sp\{v_1, \ldots, v_m\}$ is a subspace.

(2) The Null Space.

The Null space of A ($m \times n$) (denoted $N(A)$) is the set of all solutions to the homogeneous eqt $Ax = \theta$.

NOTE: $N(A) \subset \mathbb{R}^n$ (from the domain).

(1) The θ is always a sol. to the homogeneous eqt.

(2) Suppose $x, y \in N(A)$.

$\Rightarrow Ax = \theta$ & $Ay = \theta$

$\Rightarrow A(x+y) = Ax + Ay = \theta + \theta = \theta$

$\Rightarrow x + y \in N(A)$

(3) Suppose $x \in N(A)$ & $a \in \mathbb{R}$.

$\Rightarrow Ax = \theta$

$\Rightarrow A(ax) = a(Ax) = a\theta = \theta$

$\Rightarrow ax \in N(A)$.

Hence $N(A)$ is a subspace.

(3) The range or column space of A $(m \times n)$ (denoted $R(A)$) is the set of vectors defined by:

$$R(A) = \{ y \mid y = Ax \text{ for some } x \in \mathbb{R}^n \}.$$

NOTE: $R(A) \subset \mathbb{R}^m$ (from the range).

The range $R(A)$ is a subspace

(1) if $x = \theta$, then $Ax = \theta \Rightarrow \theta \in R(A)$.

(2) Suppose $y, z \in R(A) \Rightarrow \exists\, u, v$ s.t. $Au = y$ and $Av = z$.

$$\Rightarrow \quad A(u+v) = Au + Av = y + z$$

$$\cancel{\Rightarrow \exists (u+v) \text{ s.t.}}$$

$$\Rightarrow \quad \exists\, x = u + v \text{ s.t. } Ax = (y+z)$$

$$\Rightarrow \quad y + z \in R(A).$$

(3) suppose $y \in R(A)$ & $a \in \mathbb{R}$.

$$\Rightarrow \exists\, u \text{ s.t. } Au = y.$$

$$\Rightarrow A(au) = aAu = ay$$

$$\Rightarrow \exists\, x = au \text{ s.t. } Ax = ay$$

$$\Rightarrow ay \in R(A).$$

Hence $R(A)$ is a subspace.

Recall, if $A = [A_1, A_2, \dots, A_n]$, then

$$Ax = x_1 A_1 + x_2 A_2 + \dots + x_n A_n.$$

Thus the range is the linear combos of the columns ... hence the column space.

(4) The Row space

If A ($m \times n$) has rows $\vec{a}_1, \ldots, \vec{a}_m$, then the row space is $sp\{\vec{a}_1, \ldots, \vec{a}_m\}$. It is equivalent to the row space of any matrix row equiv. to A.