Matrix Inverses & Their Properties

<u>Dfn</u>: Let A be an $(n \times n)$ matrix. We say A is <u>invertible</u> if we can find an $(n \times n)$ matrix A^{-1} s.t. $A^{-1}A = AA^{-1} = I$. The matrix A^{-1} is called an <u>inverse for A</u>.

Analogy: $aa^{-1} = a^{-1}a = 1$

and $ax = b \Rightarrow a^{-1}a\,x = a^{-1}b \Rightarrow x = a^{-1}b$

$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow Ix = A^{-1}b \Rightarrow x = A^{-1}b$

<u>ex1</u>: Show $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has no inverse.

If $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ existed, $AA^{-1} = I$

$\Rightarrow a + 2c = 1$ & $2a + 4c = 0$

but this has no sol. so A^{-1} DNE.

<u>existence of ~~matrices~~ inverses</u>: which matrices are invertible?

If A $(n \times n)$ is invertible, then $\exists$ B $(n \times n)$ s.t.

$AB = I \Rightarrow A[b_1, b_2, \ldots, b_n] = [e_1, e_2, \ldots, e_n]$

$\Rightarrow [Ab_1, Ab_2, \ldots, Ab_n] = [e_1, e_2, \ldots, e_n]$

$\Rightarrow Ab_1 = e_1$; $Ab_2 = e_2$; $\ldots$; $Ab_n = e_n$. These can be solved for $\{b_1, b_2, \ldots, b_n\}$ if A is non-singular.

<u>Lemma</u>: Let $P, Q, \&\ R$ be $(n\times n)$ matrices s.t. $PQ = R$. If either P or Q is singular, so is R.

□ <u>proof</u>.

Assume $P, Q, \&\ R$ are $(n\times n)$ matrices s.t $PQ = R$.

<u>case 1</u>: suppose Q is singular.

$$PQ = R$$

since Q is singular, $\exists\ x$ s.t. $Qx = \theta$, $x \neq \theta$

$$\Rightarrow (PQ)x = Rx$$

$$\Rightarrow P(Qx) = Rx$$

$$\Rightarrow P\theta = Rx$$

$$\Rightarrow \theta = Rx$$

so Q singular $\Rightarrow$ R singular.

<u>case 2</u>: Suppose Q is nonsingular, but P singular.

$\Rightarrow \exists\ y$ s.t. $Py = \theta$, $y \neq 0$ and Q is nonsingular so $\exists\ z$ s.t. $Qz = y$.

now $PQ = R$

$$\Rightarrow (PQ)z = Rz$$

$$\Rightarrow P(Qz) = Rz$$

$$\Rightarrow Py = Rz$$

$$\Rightarrow \theta = Rz$$

so P singular $\Rightarrow$ R singular ■

<u>Thm</u>: Let A be an $(n \times n)$ matrix. Then A has an inverse iff A is nonsingular.

□ <u>proof</u>.

$(\Rightarrow)$: Assume A has an inverse.

$\Rightarrow \exists\ B$ s.t. $AB = I$.

I is nonsingular.

$\Rightarrow$ A & B are non singular (by the lemma).

what is the negation of $A \mapsto B$
A or $B \mapsto C$.

$(\Leftarrow)$: Assume A is nonsingular.

we know $\exists!\ B$ s.t. $AB = I$

<u>NTS</u>: $BA = I$ (if B commutes, it is the inverse).

since $AB = I$, B is nonsingular. $\Rightarrow \exists!\ C$ s.t. $BC = I$.

so $A = AI = A(BC) = (AB)C = IC = C$

$\Rightarrow A = C$ $\cancel{\text{\& } BA = AB}$

$\Rightarrow AB = I$ & $BA = I$

Hence B is the inverse of A & our claim is proved ■

Finding matrix inverses. (the method).

Recall that we must solve

$$Ax = e_1\ ;\ Ax = e_2\ ;\ Ax = e_3\ ;\ \cdots\ ;\ Ax = e_n.$$

which can written in the shorthand $[A \mid e_1, e_2, \ldots, e_n]$ or $[A \mid I]$.

ex 2: Find the inverse of $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 5 & 3 \end{bmatrix}$

Calculating A^{-1}.

To calculate the inverse of the nonsingular $(n \times n)$ A.

(1) Form $[A \mid I]$

(2) RREF $[A \mid I]$ to $[I \mid B]$

(3) Read $B = A^{-1}$.

Thm: A $(n \times n)$ is nonsingular iff A is row equivalent to I.

Shortcut for A (2×2).

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $\Delta = ad - bc$, then

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

note: If $\Delta = 0$, A^{-1} DNE.

Thm: Let A $(n \times n)$ be a matrix. Then the following are equiv.

(1) A is nonsingular ($Ax = \theta$ has only the trivial sol).

(2) The cols of A are L.I.

(3) $Ax = b$ always has a unique sol.

(4) A has an inverse

(5) A is row equiv. to I.

<u>Thm</u>: Let A, B be $(n \times n)$ invertible matrices.

(1) A^{-1} has an inverse & $(A^{-1})^{-1} = A$

(2) AB has an inverse & $(AB)^{-1} = B^{-1}A^{-1}$

(3) If $k \neq 0$ is a scalar, then kA has an inverse & $(kA)^{-1} = \frac{1}{k} A^{-1}$

(4) A^T has an inverse & $(A^T)^{-1} = (A^{-1})^T$.

□ proof of 3. (by Thm 8 in sec. 1.6)

Suppose k is a nonzero scalar & A $(n \times n)$ is an invertible matrix w/inverse A^{-1}.

consider $(kA)(\frac{1}{k}A^{-1}) = k(A \frac{1}{k} A^{-1})$

$= k \frac{1}{k} A A^{-1}$

$= 1 \, I$

$= I.$

If $B = \frac{1}{k} A^{-1}$

$(kA)B = k(AB)$

$= k(A \frac{1}{k} A^{-1})$

$= k \frac{1}{k} AA^{-1}$

similarly $(\frac{1}{k}A^{-1})(kA) = I.$

Hence we have shown $(kA)^{-1}$ exists & is $\frac{1}{k} A^{-1}$. ■