Linear Independence & Non-singular Matrices

recall $A = [A_1, A_2, \ldots, A_n]$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, then

$$A\vec{x} = \underbrace{x_1 A_1 + x_2 A_2 + \ldots + x_n A_n}$$

we call this a <u>linear combination</u> of the vectors $A_1, A_2, \ldots, A_n$.

So, we can write the eqt $A\vec{x} = \vec{b}$ as

$$x_1 A_1 + x_2 A_2 + \ldots + x_n A_n = \vec{b}$$

Thus $A\vec{x} = \vec{b}$ is consistent iff b is a linear combo of the columns of A.

<u>Ex1</u>: If the vectors $A_1, A_2, A_3, \vec{b}_1, \& \vec{b}_2$ are given by

$$A_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{b}_1 = \begin{bmatrix} 14 \\ 13 \\ -3 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$A = [A_1, A_2, A_3]$

solve $[A \mid b_1]$ and $[A \mid b_2] \longrightarrow$ inconsistent.

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1/4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= 1 \\ x_2 &= 3 - \tfrac{1}{4} x_3 \\ x_3 &\text{ arbitrary} \end{aligned} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$1A_1 + 2A_2 + 4A_3 = b_1$$

Notation the zero vector is written
$\vec{0}$ or θ & is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

so, the homogeneous eqt $Ax = \theta$ can be written as $x_1 A_1 + x_2 A_2 + \ldots + x_n A_n = \theta$

reminder the homogeneous eqt. always has the trivial solution $x = \theta$.

Dfn: A set of m-dimensional vectors $\{v_1, v_2, \ldots, v_n\}$ is said to be linearly independent if the only solution to $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = \theta$ is the trivial solution $a_1 = a_2 = \ldots = a_n$. The set is linearly dependent if it is not linearly ind. That is, $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = \theta$ has a non-trivial solution.

ex 2: Determine if $v_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 8 \\ 6 \\ -6 \end{bmatrix}$
are linearly independent.

$$\text{rref}\left(\left[\begin{array}{ccc|c} 2 & 4 & 8 & 0 \\ 1 & 4 & 6 & 0 \\ -3 & 0 & -6 & 0 \end{array}\right] \right) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_3$; $x_2 = -x_3$; x_3 arbitrary.

If $x_3 = 5$ (our choice), $x_1 = -10$ & $x_2 = -5$.

and $-10 v_1 - 5 v_2 + 5 v_3 = \theta$ (a non-trivial solution).

Therefore, the vectors are linearly dependent.

ex 2 revi: If $v_3 = \begin{bmatrix} 8 \\ 6 \\ 0 \end{bmatrix}$, then the vectors are L.I.

($\mathrm{rref}([v_1, v_2, v_3 \mid 0] = [I \mid 0])$).

Note: vectors are linearly dependent iff one vec. is a linear comb. of the remaining ones.

Notation: unit vectors in $\mathbb{R}^n$ are written as

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{\text{th}} \text{ position.}$$

Is it obvious that $\{e_1, e_2, \dots, e_n\}$ is ~~a~~ LI set?

ex 3: Show $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$ are linearly dependent. (simple because it is 2 eqts & 3 unknowns ... so ~~the~~ homogeneous system has non-trivial solutions).

Thm: A set of vectors that includes more vectors than entries w/in the vectors is linearly dependent. (Too many vectors).

Note: less vectors than dimensions does not guarantee L.I.

Reminder: only $(n \times n)$ systems can have a unique solution.

<u>Dfn.</u> An $(n \times n)$ matrix A is <u>nonsingular</u> if $Ax = \theta$ has only the trivial solution. Otherwise A is <u>singular</u>.

<u>Thm:</u> The $(n \times n)$ matrix $A = [A_1, A_2, \dots, A_n]$ is nonsingular iff $\{A_1, A_2, \dots, A_n\}$ is LI set.

<u>ex 4:</u> Determine whether the matrices are nonsingular.

(a) $\begin{bmatrix} 4 & 7 \\ 3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 7 \\ -8 & -14 \end{bmatrix}$

~~<u>Thm:</u> Let A be an $(n \times n)$ matrix. The eqt $Ax = b$ has a unique sol for every b iff A is nonsingular.~~

~~□ <u>proof</u>~~

~~(⇒): Suppose $Ax = b$ has a unique sol. <u>for all</u> b. Take $b = \theta$. $Ax = \theta$ has a unique sol. Therefore A is nonsingular.~~

~~(⇐): Suppose A is nonsingular & b any $(n \times 1)$ col. vec.~~

~~(A) Show $Ax = b$ has a sol.~~

~~$\{A_1, A_2, \dots, A_n, b\}$ is linearly dependent because there are too many vecs.~~

~~$\Rightarrow a_1 A_1 + a_2 A_2 + \dots + a_n A_n + a_{n+1} b = \theta$ has a non-trivial sol.~~

~~if $a_{n+1} = 0 \Rightarrow$ A is singular (contradiction). Therefore $a_{n+1} \neq 0$.~~

Thm: Let A be an $(n \times n)$ matrix. The equation $Ax = b$ has a unique solution for every b iff A is nonsingular.

□ proof.

($\Rightarrow$): Suppose $Ax = b$ has a unique solution for every b. Then it must have a unique solution when $b = \theta$. ~~That is~~ Then $Ax = \theta$ has only the trivial solution & A is nonsingular.

($\Leftarrow$): Suppose A is non-singular.

Part 1: Show $Ax = b$ has a solution and so is consistent.

consider $\{A_1, A_2, \ldots, A_n, b\}$. The set is linearly dependent by the "Too many vectors thm."

$*$ $\Rightarrow a_1 A_1 + a_2 A_2 + \ldots + a_n A_n + a_{n+1} b = \theta$ has a non trivial solution

consider a_{n+1} in particular. It is either zero or nonzero. ~~Suppose~~ Let's see if it is zero. (suppose $a_{n+1} = 0$).

$\Rightarrow a_1 A_1 + a_2 A_2 + \ldots + a_n A_n = \theta$ has a nontrivial solution which means A is singular $\Rightarrow\Leftarrow$

So $a_{n+1} \neq 0$. and solve $(*)$ for b.

$$\underbrace{-\frac{a_1}{a_{n+1}}}_{s_1} A_1 + \underbrace{-\frac{a_2}{a_{n+1}}}_{s_2} A_2 + \ldots + \underbrace{-\frac{a_n}{a_{n+1}}}_{s_n} A_n = b.$$

$\Rightarrow s_1 A_1 + s_2 A_2 + \dots + s_n A_n = b$.

Thus $Ax = b$ has a solution s given by

$$s = [s_1 \ s_2 \ \dots \ s_n]^T$$

so $Ax = b$ has a solution & is consistent when A is nonsingular.

<u>Part 2:</u> The solution to $Ax = b$ is unique (our claim)

We have one solution s to $Ax = b$. Suppose u $(n \times 1)$ is also a solution to $Ax = b$.

$\Rightarrow As = b$ and $Au = b$

$\Rightarrow As - Au = b - b$

$\Rightarrow A(s - u) = \theta$

But A is nonsingular, so $Ax = \theta$ has only the trivial solution $x = \theta$.

$\Rightarrow s - u = \theta$ and $s = u$.

Therefore, our solution is unique. ■