Algebraic Properties of Matrix Operations

One of our objectives for this course is to develop ~~an~~ understanding of mathematical abstraction & proof.

To that end, today we will ~~do~~ prove a number of properties about matrix operations.

We will need two definitions.

__Dfn__: If $A = (a_{ij})$ is an $(m \times n)$ matrix, then the __transpose__ of A, denoted A^T, is the $(n \times m)$ matrix $A^T = (b_{ij})$ where $b_{ij} = a_{ji}$ for $1 \leq j \leq m$ & $1 \leq i \leq n$.

* Rows & columns are switched.

__Dfn__: A matrix A is __symmetric__ if $A = A^T$.

* Symmetric matrices must be square.

__Note__: you should go thru Thm 7–10 & make sure all properties are in your notes.

<u>Thm</u>: For r, s scalars & A $(m \times n)$, we have $r(sA) = (rs)A$.

□ <u>proof</u>

$$
\begin{aligned}
r(sA) &= r((\underbrace{s a_{ij}}))\\
&= r(b_{ij}) \\
&= (r b_{ij}) \qquad rB = (r b_{ij})\\
&= (\underbrace{rs}\, a_{ij})\\
&= (t\, a_{ij}) \\
&= tA \qquad \text{by dfn.}\\
&= (rs)A \quad \blacksquare
\end{aligned}
$$

Thm: For A, B $(m \times n)$ matrices, $(A+B)^T = A^T + B^T$.

□ proof. Note the dimensions match. (unnecessary)

consider $(A+B)^T = C^T$ where $C = A + B$

$$= (c_{ij})^T$$

$$= (c_{ji})$$

$$= (a_{ji} + b_{ji})$$

$$= (a_{ji}) + (b_{ji})$$

$$= A^T + B^T \quad \blacksquare$$

<u>Thm</u>: For any $(m \times n)$ Q, QQ^T is symmetric.

<u>NTS</u>: $(QQ^T)^T = QQ^T$

□ <u>proof</u>. Note: the dimensions match

consider $((QQ^T)^T)_{ij} = (QQ^T)_{ji}$

$$= \sum_{k=1}^{n} q_{jk} (Q^T)_{ki}$$

$$= \sum_{k=1}^{n} q_{jk} \cdot q_{ik}$$

$$= \sum_{k=1}^{n} q_{ik} \cdot q_{jk}$$

$$= \sum_{k=1}^{n} q_{ik} \cdot (Q^T)_{kj}$$

$$= (QQ^T)_{ij} \quad \blacksquare$$

<u>Thm</u>: For A $(m \times n)$ and B, C $(n \times p)$, the equality $A(B+C) = AB + AC$ holds.

□ <u>proof.</u> Note the dimensions match.

consider $A(B+C) = AD$ where $B + C = D$.

The ij^{th} entry $(AD)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot d_{kj}$

$$= \sum_{k=1}^{n} a_{ik} \cdot (b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj})$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

$$= (AB)_{ij} + (AC)_{ij}$$

Thus $AD = AB + AC$ & our claim is proved ∎

<u>Thm</u>: If A is $(m \times n)$ & C is $(n \times p)$, then

$$(A\,C)^T = C^T A^T.$$

□ <u>proof</u>. Note the dimensions match.

Consider

$$\left((A\,C)^T\right)_{ij} = (AC)_{ji}$$
$$= \sum_{k=1}^{n} a_{jk}\, c_{ki}$$
$$= \sum_{k=1}^{n} c_{ki}\, a_{jk}$$
$$= \sum_{k=1}^{n} (C)_{ki}\, (A)_{jk}$$
$$= \sum (C^T)_{\cancel{ki}\,ik}\, (A^T)_{kj}$$
$$= C^T A^T$$

Therefore $(A\,C)^T = C^T A^T$.

Thm: There exists a unique $(m \times n)$ matrix Θ s.t. $A + \Theta = A$ $\forall$ $(m \times n)$ A.

$\square$ proof.

Part 1: show that $\exists$ B s.t. $A + B = A$.

Let A be any $(m \times n)$ matrix and let B be an $(m \times n)$ matrix s.t. $b_{ij} = 0$.

Now

$$\begin{aligned}(A+B)_{ij} &= a_{ij} + b_{ij} \\ &= a_{ij} + 0 \\ &= a_{ij} \\ &= (A)_{ij}\end{aligned}$$

so $A + B = A$. That is, a B exists.

Part 2: Show B is unique.

Let A be any $(m \times n)$ matrix & $B \neq C$ $(m \times n)$ to be such that:

$A + B = A$ and $A + C = A$ ~~and B ≠ C~~.

$\Rightarrow A + B = A + C$

$\Rightarrow (A+B)_{ij} = (A+C)_{ij}$

$\Rightarrow a_{ij} + b_{ij} = a_{ij} + c_{ij}$

$\Rightarrow b_{ij} = c_{ij}$

$\Rightarrow B = C$ $\Rightarrow\Leftarrow$

Therefore B exists & is unique. We call this matrix Θ, the zero matrix. ■