Matrix Operations

Dfn Matrices A & B are equal provided they have the same dimension $(m \times n)$ & $a_{ij} = b_{ij}$ for all $1 \le i \le m$ & $1 \le j \le n$.

Notation Let $(Q)_{ij}$ be the ij^{th} entry of Q.

Dfn Let $A = (a_{ij})$ & $B = (b_{ij})$ be matrices of the same size. The sum $A + B$ is:

$$(A+B)_{ij} = a_{ij} + b_{ij}$$

(add components).

Dfn Let $A = (a_{ij})$ & let r be a scalar. The scalar product rA is defined by $(rA)_{ij} = r a_{ij}$

Note: Vectors can be written as vectors.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$ is an n-dimensional vector.

ex 1: Give the vector form for the general solution to

$$[A|b] = \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 6 & 11 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + x_3 \\ 11 - 6x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \\ 1 \end{bmatrix}$$

$x_1 = 4 + 3x_3$; $x_2 = 11 - 6x_3$; x_3 arbitrary

<u>matrix multiplication</u> $A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$

where the i^{th} row of A is multiplied by the j^{th} column of B & the result is c_{ij}

<u>ex2</u>:

$$\overset{A}{\begin{bmatrix} 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \\ 5 & 10 \end{bmatrix}}_{5\times 2} \cdot \overset{B}{\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 19 & 12 & 5 \\ 23 & 14 & 5 \\ 27 & 16 & 5 \\ 31 & 18 & 5 \\ 35 & 20 & 5 \end{bmatrix}_{5\times 3}$$

match

<u>Dfn</u>: Let $A = (a_{ij})$ be an $(m\times n)$ matrix & $B = (b_{ij})$ be $(r\times s)$. If $n = r$ then the <u>product</u> AB is $(m\times s)$ & defined by:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

If $n \neq k$, then AB is not defined.

<u>ex 3</u> compare

a) $\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

b) $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix}$

* matrix multiplication is not commutative.

$A \cdot B \neq B \cdot A$ (in general).

matrix mult. provides a concise notation.

ex4:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{b}$$

so, we have found a connection between

$$[A|b] \quad \text{and} \quad Ax = b$$

We will use this relationship a lot because it provides a way to solve equations thru row reduction.

Alternate formulations of matrix mult.

(1) notation: the $(m \times n)$ matrix $A = (a_{ij})$ can be written

$$A = [A_1, A_2, \dots, A_n]$$

where A_j is the jth column of A.

Thm: $Ax = x_1A_1 + x_2A_2 + \dots + x_nA_n$

the proof is left for you in $\boxed{\#4}$

1,5
4/5

ex5: If $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ & $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then $AX = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 2x_1 + 5x_2 \\ 3x_1 + 6x_2 \end{bmatrix}$

$$= x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Thm: Let A be an $(m \times n)$ matrix, and let $B = [B_1, B_2, \ldots, B_s]$ be an $(n \times s)$ matrix whose kth col. is B_k, then the jth col of AB is AB_j so

$$AB = [AB_1, AB_2, \ldots, AB_s]$$

□ proof. If $A = (a_{ij})$ & $B = (b_{ij})$, then the jth col. of AB ; ~~AB_j where~~ ~~$B_j = [b_{1j}, b_{2j}, \ldots, b_{nj}]^T$ (column vec)~~, is

$$(AB)_j = \begin{bmatrix} \sum a_{1k} b_{kj} \\ \sum a_{2k} b_{kj} \\ \vdots \\ \sum a_{nk} b_{kj} \end{bmatrix} = AB_j \text{ where } B_j = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

$\boxed{\begin{matrix}1.5\\5/5\end{matrix}}$

<u>exbi</u> Let A & B be as in (ex 2).

we already found AB, now, find

$$\left[AB_1 \quad AB_2 \quad AB_3\right]$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix}19\\23\\27\\31\\35\end{bmatrix} \quad \begin{bmatrix}12\\14\\16\\18\\20\end{bmatrix} \quad \begin{bmatrix}5\\5\\5\\5\\5\end{bmatrix}$$