Consistent Systems of Linear Equations

a sys. of lin. eqts $\Big\langle$ unique sol. / ∞ sol. / no sol.

can we identify which case or eliminate a possibility w/o fully solving the sys?

note: an $(m \times n)$ system has a $[m \times (n+1)]$ augmented matrix associated w/ it.

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 3 \end{aligned}$$

is a (2×3) sys. w/ a (2×4) aug. matrix.

consider the sys. w/ aug. matrix

$$[A|b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & & & & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & & \cdots & a_{mn} & b_m \end{bmatrix}$$

$[A|b]$ is row equivalent to $[C|d]$ which is in RREF.

note: The sys is inconsistent iff $[C|d]$ has a row $[0, 0, \ldots, 0, 1]$.

1.3
2/5

Note! Every variable corresponding to a leading 1 in [c|d] is a dependent variable.

ex!

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 4 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 6 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dependent variables $\begin{cases} x_1 = 9 - 3x_3 - 4x_5 \cancel{-5x_6} \\ x_4 = 3 - 6x_5 \cancel{-5x_6} \\ x_6 = 2/4 \end{cases}$

ind. var. $\{$ x_2, x_3, x_5, x_6 arbitrary

Let r be the number of non-zero rows in [c|d].

r is the rank of C.

Note: $r \leq n+1$

Note: If the sys. represented by [c|d] is consistent, then $r \leq n$.

$\Rightarrow$ r dependent variables & $(n-r)$ independent.

Thm: Let $[C|d]$ be an $[m \times (n+1)]$ matrix in RREF where $[C|d]$ represents a consistent sys. then $r \leq n$ & in the sol. to the sys there are $(n-r)$ variables that can be assigned arb. values.

ex 1 rev: $[C|d]$ represents a consistent sys.

the matrix has 3 nonzero rows
$\Rightarrow r = 3$ (the rank)

$\Rightarrow$ there are $n - r = 7 - 3 = 4$
ind. variables & 3 dependent.

The Thm shows us that the only possibilities for the sys. represented by $[C|d]$ are:

(1) the sys. is inconsistent.

(2) sys. consistent & $r < n$.
$\Rightarrow$ $n-r$ unconstrained variables.

(3) sys. consistent & $r = n$. no unconstrained variables & hence a unique sol.

Corollary: consider an $(m \times n)$ system of lin. eqs. If $m < n$, then either the sys. is inconsistent or it has infinitely many sol

$\begin{bmatrix} 1.3 \\ 4/5 \end{bmatrix}$

ex 2: (a) What are the possibilities for the solution set of a (5×8) sys. of linear equations?

$\Rightarrow$ no sol. or ∞ many sol.

(b) What are the possible number of ind. variables?

$\Rightarrow$ 3, 4, 5, ..., 8 are the possible # ind. variables $(n-r)$. since $r \leq 5$, r must be $1, 2, ..., 5$

note: $r = 0$ ... the trivial sys.

What are the possibilities for the sys.

ex 3: $3x_1 + 2x_2 + x_3 = 0$

$x_1 + 2x_2 + 3x_3 = 0$

$m < n$, so by con. we have

(i) no sol.

(ii) ∞ sol.

The trivial sol. exists, therefore ∞ sol.

The previous example shows a homogeneous sys. of eqts.

$\boxed{\begin{matrix}1.3\\5/5\end{matrix}}$

<u>Def</u>: A homogeneous sys. of lin. eqs is of the form:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

The trivial solution always exists.

$\Rightarrow$ homogeneous sys. are always consistent.

<u>thm</u>: A homogeneous ($m \times n$) sys. of lin. eqts. always has infinitely many nontrivial solutions when $m < n$.

<u>ex3</u> Now row reduce (ex3).

<u>ex4</u>: What are the possible solution sets for:

$$x_1 + 2x_2 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$4x_3 = 0$$

<u>ex5</u>: Find the eq. for the conic section of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ thru $(-4,1), (-1,2), (3,2), (5,1), (7,-1)$