Mult. of Radicals of Neg #s

Wednesday, June 06, 2007
9:01 PM

Claim: $\sqrt{-2}\cdot\sqrt{-3}$ must be calculated via $i\sqrt{2}\cdot i\sqrt{3}$ to get $-\sqrt{6}$, not $\sqrt{6}$.

Why?

First examine why this property is true for nonnegative #s.

What does $\sqrt{x}$ mean?

(How do we know there's only one?)

$\sqrt{x}$ is the single nonnegative number that makes $(\quad)^2 = x$ true. ($\sqrt{9}=3$ because $(3)^2=9$ and $3 \geq 0$)

$\sqrt{x}$ is the "principal square root" (PSR)

So $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$ means the (PSR of a)(PSR of b) = (PSR of ab)

Prove that for $a, b \geq 0$, $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

Proof: To show this, we use the fact that $\sqrt{ab}$ is a single, nonnegative number that satisfies $(\quad)^2 = ab$. So anything that meets these conditions must equal $\sqrt{ab}$.

$\sqrt{a}\cdot\sqrt{b}$ is nonnegative since $\sqrt{a} \geq 0$ and $\sqrt{b} \geq 0$

$(\sqrt{a}\cdot\sqrt{b})^2 = ab$ since

$$
\begin{aligned}
(\sqrt{a}\cdot\sqrt{b})^2 &= (\sqrt{a}\cdot\sqrt{b})(\sqrt{a}\cdot\sqrt{b}) && \text{by definition} \\
&= \sqrt{a}\cdot\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{b} && \text{by commut. law} \\
&= (\sqrt{a})^2\cdot(\sqrt{b})^2 && \text{by definition} \\
&= a\cdot b && \text{by definition of } \sqrt{a}, \sqrt{b}
\end{aligned}
$$

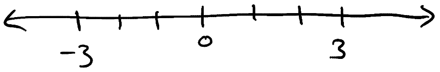
Thus $\sqrt{a}\cdot\sqrt{b}$ meets both conditions, Thus $\sqrt{a}\cdot\sqrt{b}$ meets both conditions, proving $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$. QED

What about when a or $b<0$, and specifically, $a<0$ and $b<0$?

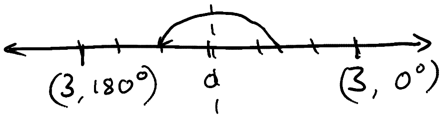
$\sqrt{a}$ means what if $a<0$? Since no real number will solve $(\ )^2=a$ when a is negative, we must expand the number system to find solutions.

Since negative #s are simply distances in "another direction" from zero, we'll use angles instead of the symbol "-"

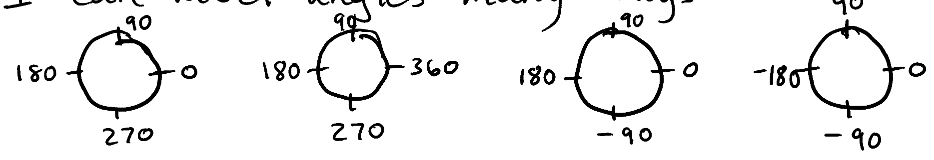
Old:



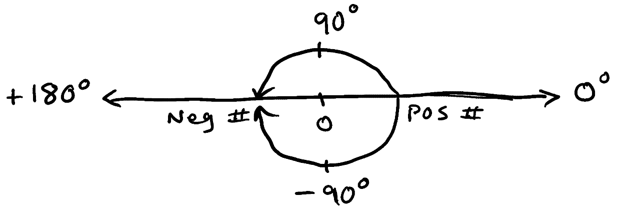
New:



We're describing #s as distances in a direction compared to the positive axis (0°). But I can label angles many ways



so we standardize this:



$-180<\text{Angle}\leq 180°$

The number "-3" is now shown as $3e^{180i}$ (the reason for the e^{180i} form will not be explained here). Since -180°, 180°, 540°, 900°, etc.

are all the same direction (negative x-axis), the angle that falls in the diagram above is called the "principal value of the argument" (angle)

In this context, square roots can be interpreted using the fractional power definition:

$$\sqrt{-3} = \sqrt{3e^{180i}} = \left(3e^{180i}\right)^{1/2} = \underbrace{3^{1/2} e^{90i} = \sqrt{3}}_{\text{use principal square root}} e^{90i}$$

Just like with positive numbers, any number will have two square roots, so $\sqrt{}$ still refers to the principal square root — the square root having an angle which is half the principal angle of the radicand.

Example using -3:

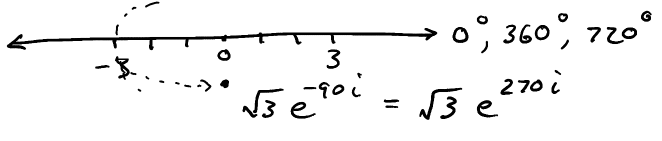
$$-3 = 3e^{-180i} = \underset{\substack{\uparrow \\ \text{principal value}}}{3e^{180i}} = 3e^{540i} = 3e^{900i} = \cdots$$

Using ½ powers on each of these definitions as a way to find square roots gives:

same location

$$(-3)^{1/2} = \sqrt{3}e^{-90i} = \sqrt{3}e^{90i} = \sqrt{3}e^{270i} = \sqrt{3}e^{450i} = \cdots$$

same location



and since 180° is the principal value for −3's angle, then 90° is the principal value for $\sqrt{-3}$'s angle. This means

$$-3 = 3e^{180i}, \text{ so } \underbrace{\sqrt{-3}}_{\text{principal square root of } -3} = 3e^{90i}$$

principal square root of -3

Now we can try to prove $\sqrt{a}\cdot\sqrt{b} = -\sqrt{ab}$ when $a, b < 0$.

Proof: If $a, b < 0$, then $a = |a|e^{180i}$, $b = |b|e^{180i}$.
Since $ab > 0$, its square root is "normal" → $\sqrt{ab}$ is the positive number for which $(\ \ )^2 = ab$.
The numbers $|a|$ and $|b|$ are positive numbers, so they have "normal" principal square roots, $\sqrt{|a|}$, $\sqrt{|b|}$. By our earlier proof for positive numbers,

$$\sqrt{|a|}\cdot\sqrt{|b|} = \sqrt{|a|\cdot|b|}$$

Because it's always true that $|a|\cdot|b| = |a\cdot b|$, and $a\cdot b > 0$, then we can write

$$\sqrt{|a|}\cdot\sqrt{|b|} = \sqrt{|ab|} = \sqrt{ab}.$$

Using the explanation of $\sqrt{\ }$ for negative numbers,

principal angle/$\sqrt{\ }$

$$\begin{aligned}
\sqrt{a}\cdot\sqrt{b} &= \sqrt{|a|}\,e^{90i}\cdot\sqrt{|b|}\,e^{90i} && \text{(Translate to angle description)}\\
&= \sqrt{|a|}\cdot\sqrt{|b|}\,e^{90i}e^{90i} && \text{(commut.)}\\
&= \sqrt{ab}\,e^{90i}\cdot e^{90i} && \text{(last paragraph)}\\
&= \sqrt{ab}\,e^{(90+90)i} && \text{(exp. rules)}\\
&= \sqrt{ab}\,e^{180i} && \text{Principal angle}\\
&= -\sqrt{ab} && \text{(Translate from angle description)}
\end{aligned}$$

Thus for $a, b < 0$, $\sqrt{a}\cdot\sqrt{b} = -\sqrt{ab}$. QED.

But it seems so natural to say $\sqrt{-2}\cdot\sqrt{-7} = \sqrt{14}$, so what's being violated?

$$\begin{aligned}
\sqrt{-2} &= \left(2e^{180i}\right)^{1/2} = \sqrt{2}\,e^{90i}\\
\sqrt{-7} &= \left(7\cdot e^{180i}\right)^{1/2} = \sqrt{7}\,e^{90i}
\end{aligned}$$

Principal square roots/angles

$$\sqrt{14} = \Big(14\underset{\substack{\uparrow \\ \text{principal angle}}}{e^{0i}}\Big)^{1/2} = \sqrt{14}\, e^{0i} \quad \leftarrow \text{Principal square root/angle}$$

$$\text{But } \sqrt{2}\, e^{90i} \cdot \sqrt{7}\, e^{90i} = \sqrt{2 \cdot 7}\, e^{180i} \neq \sqrt{14}\, e^{0i}$$

It's frustrating that $\sqrt{-2} \cdot \sqrt{-7} \neq \sqrt{14}$ because it seems so natural! Fortunately, the broader mathematical theory agrees it <u>is</u> true that

$$\text{Square root of }(-2) \cdot \text{square root of }(-7) = \text{square root of }(14)$$

The key difference is not restricting to the <u>principal</u> square roots. Recall,

$$-2 = 2e^{-180i} = 2e^{180i} = 2e^{540i} = 2e^{900i} = \text{etc.}$$

so

$$(-2)^{1/2} = \sqrt{2}\, e^{-90i} = \sqrt{2}\, e^{90i} = \sqrt{2}\, e^{270i} = \sqrt{2}\, e^{450i} = \text{etc.}$$

which breaks into two families:

$$\sqrt{2}\, e^{-90i} = \sqrt{2}\, e^{270i} = \sqrt{2}\, e^{(-90+360n)i} \qquad n \in \mathbb{Z}$$

$$\text{and } \sqrt{2}\, e^{90i} = \sqrt{2}\, e^{450i} = \sqrt{2}\, e^{(90+360n)i} \qquad n \in \mathbb{Z}$$

If we do this for -7 as well and just focus on the ±90° choices, we get

$$\begin{aligned}
(-2)^{1/2}(-7)^{1/2} &= \sqrt{2}\, e^{-90i} \sqrt{7}\, e^{-90i} = \sqrt{14}\, e^{-180i} = -(14)^{1/2} \\
&= \sqrt{2}\, e^{90i} \sqrt{7}\, e^{-90i} = \sqrt{14}\, e^{0i} \\
&= \sqrt{2}\, e^{-90i} \sqrt{7}\, e^{90i} = \sqrt{14}\, e^{0i} \\
&= \sqrt{2}\, e^{90i} \sqrt{7}\, e^{90i} = \sqrt{14}\, e^{180i} = -(14)^{1/2}
\end{aligned}$$

$$\left.\begin{matrix} \sqrt{14}\, e^{0i} \\ \sqrt{14}\, e^{0i} \end{matrix}\right\} = (14)^{1/2}$$

Thus, depending on the choices of square roots, we get

$(-2)^{1/2}(-7)^{1/2} = (14)^{1/2}$, the "intuitive" answer

or $(-2)^{1/2}(-7)^{1/2} = -(14)^{1/2}$, the answer from the principal square roots.