16 SAMPLE EXAM

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. Match up each entry in the first column to one in the second. A given entry in the second column can be used once, more than once, or not at all.

If a vector field **F** is the gradient of some scalar function, then **F** is ____________ .	conservative
If a curve C is the union of a finite number of smooth curves, then C is ____________ .	curl
If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is ____________ in D.	divergence
If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, **F** is defined everywhere in $\mathbb{R}^2$ and $\partial P/\partial y = \partial Q/\partial x$, then **F** is ____________ .	flux
If a curve C doesn't intersect itself anywhere between its endpoints, then C is ____________ .	irrotational
If **F** is a vector field on $\mathbb{R}^3$ then $\nabla \times \mathbf{F}$ is called the ____________ .	path independent
If **F** is a vector field on $\mathbb{R}^3$ then $\nabla \cdot \mathbf{F}$ is called the ____________ .	piecewise smooth
If **F** is a continuous vector field defined on an oriented surface S then $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is the ____________ .	simple
If **F** is a vector field and curl $\mathbf{F} = 0$ at a point P, then **F** is ____________ at P.	simply-connected

2. Consider the oriented surface S for $z \geq 0$, consisting of the portion of the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane and with outward normal.

(a) What is the boundary curve $C = \partial S$ and what direction is its positive orientation?

(b) What surface S_1 and what assignment of a normal in the xy-plane has the same boundary curve $C = \partial S_1$ with the same orientation?

(c) Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, if $\mathbf{F} = (xe^z - 3y)\,\mathbf{i} + \left(ye^{z^2} + 2x\right)\mathbf{j} + (x^2y^2z^3)\,\mathbf{k}$.

3. Parametrize the boundary curve $C = dS$ of the surface S: $\dfrac{x^2}{9} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$, $z \leq 0$, so that it has positive orientation with respect to S.

4. (a) Find a counterclockwise parametrization of the ellipse $x^2 + \frac{y^2}{4} = 1$.

(b) Compute the double integral

$$\iint_{0 \le x^2+y^2/4 \le 1} 3x^2 y \, dA$$

Hint: Can you find a vector function $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^3 y$?

5. Consider $\mathbf{F}(x, y, z) = -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j} + z\,\mathbf{k}$.

(a) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle $x^2 + y^2 = 1$ in the xy-plane, oriented counterclockwise.

(b) Show that curl $\mathbf{F} = \langle 0, 0, 0 \rangle$ everywhere that $\mathbf{F}$ is defined.

(c) Indicate why you cannot use Stokes' Theorem on this problem. [That is, explain why your answers to (a) and (b) don't contradict one another.]

6. (a) Use the Divergence Theorem to show that, for a closed surface S with an outward normal which encloses a solid region B,

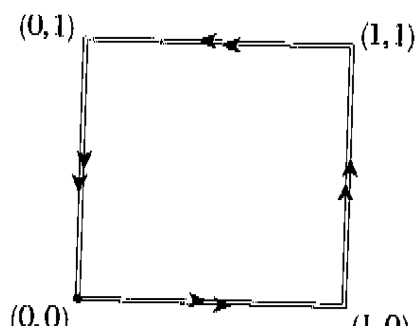
$$\text{Volume}(B) = \iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = \langle x, 0, 0 \rangle$.

(b) Use part (a) to show that the volume enclosed by the unit sphere is $\frac{4}{3}\pi$.

(c) Compute $\iint \mathbf{F} \cdot d\mathbf{S}$ if $\mathbf{F}(x, y, z) = \langle 3x, 4y, 5z \rangle$.

7. Compute the work done by the vector field $\mathbf{F}(x, y) = (\sin x + xy^2)\,\mathbf{i} + \left(e^y + \frac{1}{2}x^2\right)\mathbf{j}$ in $\mathbb{R}^2$, where C is the path that goes around the unit square twice.



8. Consider the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$.

(a) Compute curl $\mathbf{F}$.

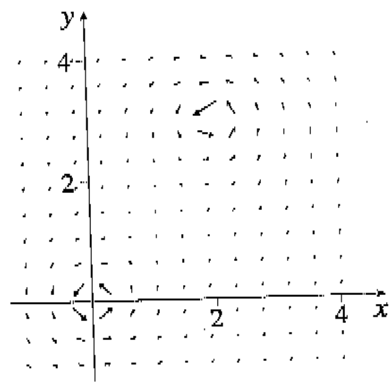
(b) If C is any path from $(0, 0, 0)$ to (a_1, a_2, a_3) and $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{a} \cdot \mathbf{a}$.

9. Consider the vector fields $\mathbf{F}(x, y) = \frac{-y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$ and

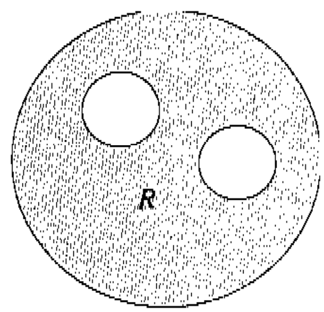
$$\mathbf{G}(x, y) = \frac{-(y-3)}{(x-2)^2 + (y-3)^2}\,\mathbf{i} + \frac{x-2}{(x-2)^2+(y-3)^2}\,\mathbf{j}.$$

(a) Given that curl $\mathbf{F}(x, y) = \mathbf{0}$ for $(x, y) \neq (0, 0)$, compute curl $\mathbf{G}(x, y)$ for $(x, y) \neq (2, 3)$.

(b) Below is the plot of the vector field $\mathbf{F}(x, y) + \mathbf{G}(x, y)$. Describe where this vector field is defined. Describe where it is irrotational.



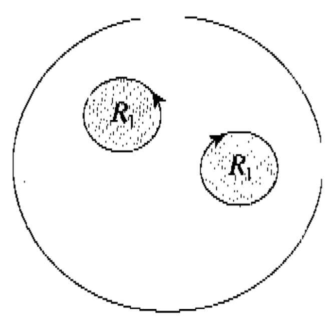
10. Consider the shaded region below.



(a) Draw arrows on the boundaries ∂R of R to give it a positive orientation.

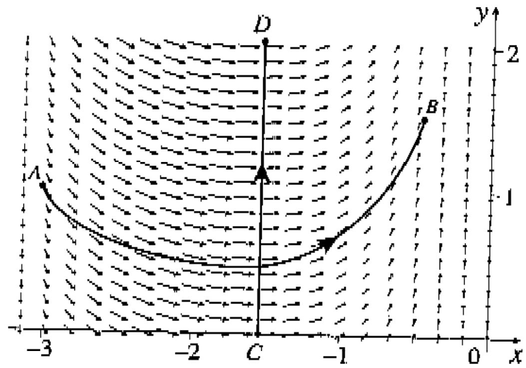
(b) If the outer circle has radius 4 and the two smaller circles have radius 1, evaluate $\frac{1}{2}\left(\int_{\partial R} y\,dx - x\,dy\right)$.

(c) Compute $\frac{1}{2}\left(\int_{\partial R_1} y\,dx - x\,dy\right)$, where R_1 is the new shaded region in the figure below. Each smaller circle has radius 1.



11. Show that the surface parametrization given by $\mathbf{r}(s, t) = \left\langle 2\cos t \sin s, \sin t \sin s, \frac{1}{\sqrt{2}}\cos s \right\rangle$, where $0 \le t \le 2\pi$, $0 \le s \le \pi$, describes the ellipsoid $\frac{1}{4}x^2 + y^2 + 2z^2 = 1$.

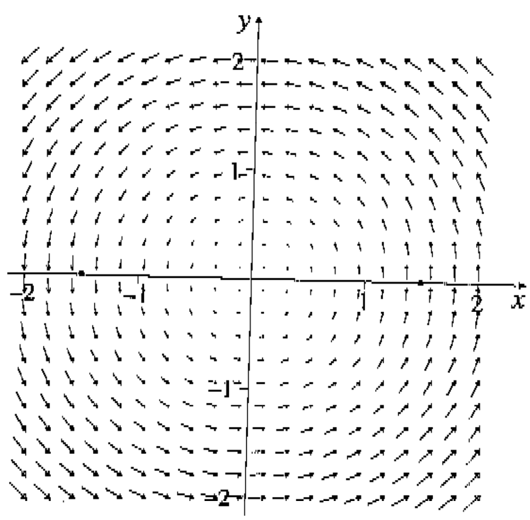
12. Consider the following vector field $\mathbf{F}$.



(a) Is the line integral of $\mathbf{F}$ along the path from A to B positive, negative, or zero? How do you know?

(b) Is the line integral of $\mathbf{F}$ along the path from C to D positive, negative, or zero? How do you know?

13. Consider the vector field below.

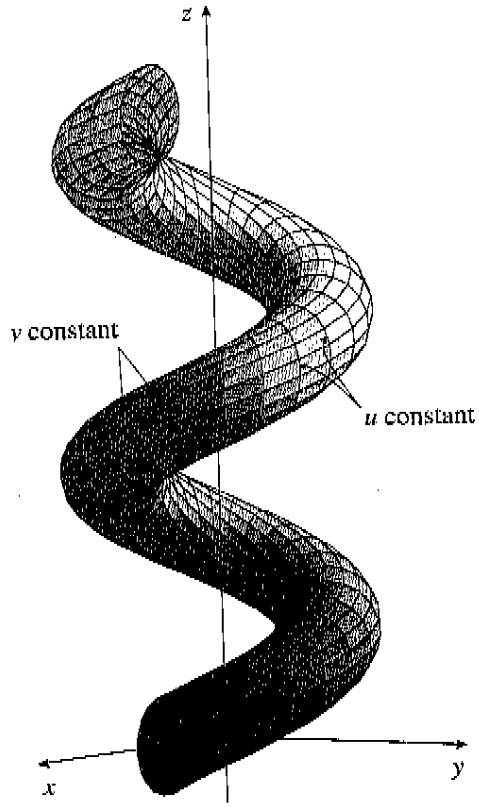


(a) Draw and label a curve C_1 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} > 0$.

(b) Draw and label a curve C_2 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_2} \mathbf{F} \cdot d\mathbf{s} < 0$.

(c) Draw and label a curve C_3 from $(-1.5, 0)$ to $(1.5, 0)$ such that $\int_{C_3} \mathbf{F} \cdot d\mathbf{s} \approx 0$.

14. The following parametric surface has grid curves which can be shown to be circles when u is constant.



$$x = (2 + \sin v)\cos u \qquad y = (2 + \sin v)\sin u \qquad z = u + \cos v$$

(a) Find the center and radius of the circle at $u = \frac{\pi}{2}$.

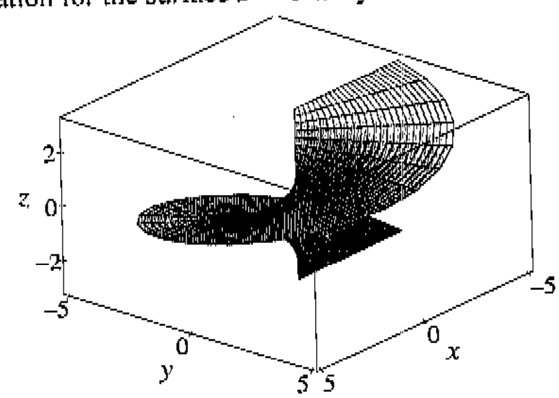
(b) Find the normal vector to S at the point P generated when $u = v = \frac{\pi}{2}$.

15. Find the equations for the following parametrized surfaces in rectangular coordinates, and describe them in words.

(a) $\left\langle t, \sqrt{1 - t^2}\sin s, \sqrt{1 - t^2}\cos s\right\rangle$

(b) $\left\langle t^2, s^2, s^2 + t^2\right\rangle$

16. Find a parametric representation for the surface $z = \theta$ in cylindrical coordinates.



17. Consider the surfaces $S_1 : \frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{4} = 1, z \geq 0$ and $S_2 : 4z = 9 - x^2 - y^2, z \geq 0$. Let $\mathbf{F}$ be any vector field with continuous partial derivatives defined everywhere. Show that $\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

18. Set up and evaluate the integral for the surface area of the parametrized surface

$$x = u + v \qquad y = u - v \qquad z = 2u + 3v$$
$$0 \leq u \leq 1 \qquad 0 \leq v \leq 1$$