14 SAMPLE EXAM

Problems marked with an asterisk (*) are particularly challenging and should be given careful consideration.

1. (a) Consider the function $f(x,y)=\dfrac{1}{x^2+y^2+1}$. Find equations for the following level surfaces for f, and sketch them.

(i) $f(x,y)=\frac{1}{5}$

(ii) $f(x,y)=\frac{1}{10}$

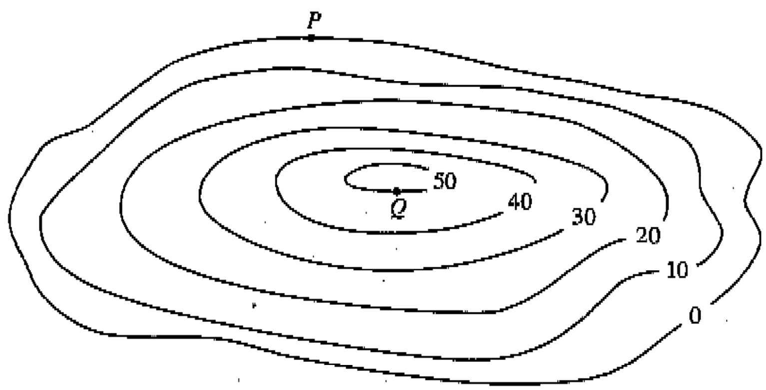
(b) Find k such that the level surface $f(x,y)=k$ consists of a single point.

(c) Why is k the global maximum of $f(x,y)$?

2. Is the function $f(x,y)=\sin^2(xy^2)$ a solution to the partial differential equation $\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}=(2x+y)(2y)\cos(xy^2)\sqrt{f}$ when $\sin(xy^2)\geq 0$?

3. Is it possible to find a function for which it is true that, for all $x>0$ and $y>0$, $f_x>0$ and $f_y<0$, and $f(x,y)>0$? If so, give an example. If not, why not?

4.



The above is a topographical map of a hill.

(a) Starting at P, sketch the path of steepest ascent to the peak elevation of 50 yards.

(b) Suppose it rains, and water runs down the hill starting at Q. At what point would you expect the water to reach the bottom? Justify your answer.

14.8 **5.** ~~Find the absolute maximum and minimum of $f(x,y)=x^2+xy+y^2$ on the disk $\{(x,y)\mid x^2+y^2\leq 9\}$.~~

6. Consider the ellipsoid $\dfrac{x^2}{4}+2z^2+\dfrac{y^2}{4}=1$. Using geometric reasoning or otherwise, find the equation of the tangent plane at

(a) $\left(\sqrt{2},\sqrt{2},0\right)$.

(b) $\left(0,0,\frac{1}{\sqrt{2}}\right)$.

7. Describe the level surfaces $f(x,y,z)=k$ for the function $f(x,y,z)=1-x^2-\dfrac{y^2}{2}-\dfrac{z^2}{3}$ and the values $k=-1$, $k=1$, and $k=2$.

8. Suppose that the amount of energy $F(x, y, z)$ emanating from a source at $(0, 0, 0)$ is inversely proportional to one more than the square of the distance from the origin measured only in the the xy-plane, and is directly proportional to the height above the xy-plane. Assume that all of the constants of proportionality are equal to 1.

(a) What is an equation for the energy as a function of x, y, and z?

(b) Where is there no energy at all?

(c) Sketch the level surface $F(x, y, z) = 1$.

9. Consider the function

$$f(x, y) = \frac{x + y}{|x| + |y|}$$

(a) Evaluate the following

(i) $f(1, 1)$

(ii) $f(1, -1)$

(iii) $f(-1, 1)$

(iv) $f(-1, -1)$

14.2 (b) Does this function have a limit at $(0, 0)$?

14.2 **10.** Consider the function

$$f(x, y) = \begin{cases} \dfrac{2x^2 + 3y^2}{x - y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) Compute $f_x(0, 0)$ directly from the limit definition of a partial derivative

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

(b) Compute $f_y(0, 0)$.

11. If $f(0, 0) = 0$, $f_x(0, 0) = 0$, $f_y(0, 0) = 0$, and $f(x, y)$ is differentiable at $(0, 0)$, does this imply that $f(x, y) = 0$ for some point $(x, y) \neq (0, 0)$? Justify your result, or give a counterexample.

12. Consider the sphere $x^2 + y^2 + z^2 = 9$. Find the equation of the plane tangent to this sphere at

(a) $(3, 0, 0)$.

(b) $(2, 2, 1)$.

13. Suppose that $f(x, y) = e^{x-y}$ and $f(\ln 2, \ln 2) = 1$. Use the technique of linear approximation to estimate $f(\ln 2 + 0.1, \ln 2 + 0.04)$.

14. Let $g(u)$ be a differentiable function and let $f(x, y) = g(x^2 + y^2)$.

(a) Show that $y f_x = x f_y$.

(b) Find the direction of maximal increase of f at $(1, 1)$ in terms of g'.

15. Let f be a function of two variables with the following properties:

- $\dfrac{\partial f}{\partial x}$ is defined near $(0, 0)$, continuous at $(0, 0)$ and $\dfrac{\partial f}{\partial x}(0, 0) = 0$
- $\dfrac{\partial f}{\partial y}$ is defined near $(0, 0)$, continuous at $(0, 0)$ and $\dfrac{\partial f}{\partial y}(0, 0) = 0$
- $\dfrac{\partial^2 f}{\partial x \partial y}(0, 0) = 1$
- $\dfrac{\partial^2 f}{\partial y \partial x}(0, 0) = -1$

Answer true or false to the following, and give reasons for your answers.

(a) f is differentiable at $(0, 0)$.

(b) There is a horizontal plane that is tangent to the graph of f at $(0, 0)$.

(c) The functions $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ are both continuous at $(0, 0)$.

(d) The linear approximation to $f(x, y)$ at $(0, 0)$ is $L(x, y) = x - y$.

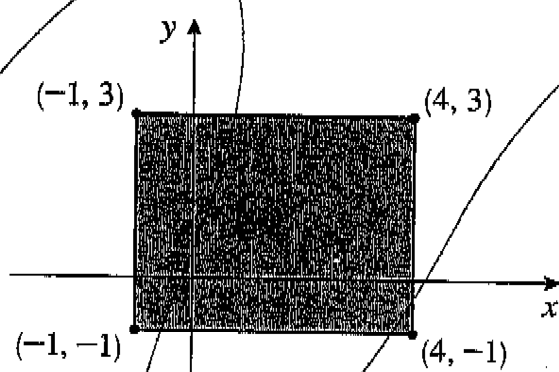
16. Suppose $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_{\mathbf{u}}(f(a, b)) = 3$ and $D_{\mathbf{v}}(f(a, b)) = \sqrt{2}$.

(a) Find $\nabla f(a, b)$.

(b) What is the maximum possible value of $D_{\mathbf{w}}(f(a, b))$ for any $\mathbf{w}$?

(c) Find a unit vector $\mathbf{w} = \langle w_1, w_2 \rangle$ such that $D_{\mathbf{w}}(f(a, b)) = 0$.

14.7 **17.** Let $f(x, y) = e^{-(x^2+y^2)}$. Find the maximum and minimum values of f on the rectangle shown below. Justify your answer.



14.8 **18.** Which point on the surface $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$ is closest to the origin?