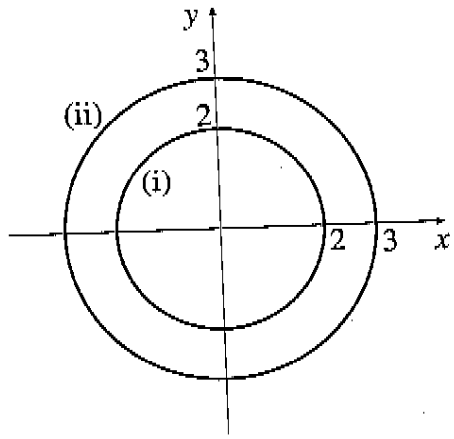
14 SAMPLE EXAM SOLUTIONS

1. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

(a) (i) $f(x, y) = \frac{1}{5} \Rightarrow 5 = x^2 + y^2 + 1$
$\Rightarrow x^2 + y^2 = 4$

(ii) $f(x, y) = \frac{1}{10} \Rightarrow x^2 + y^2 = 9$



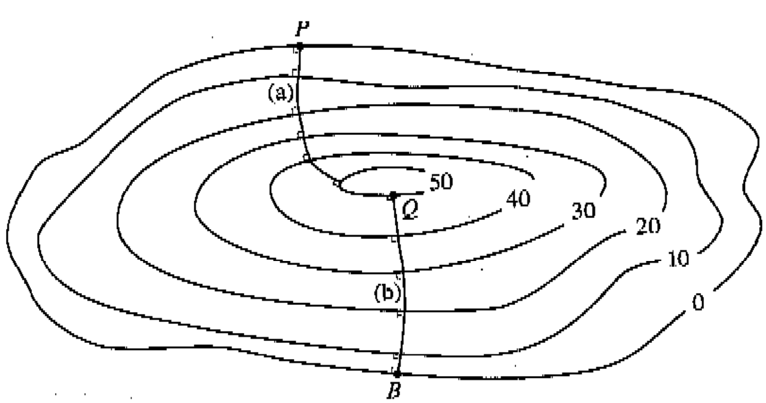
(b) $f(x, y) = 1$ consists of a single point $(0, 0)$. Otherwise, $k < 1$ always gives the circle $x^2 + y^2 = 1 - 1/k$.

(c) $\dfrac{1}{x^2 + y^2 + 1} \leq 1$ for any point (x, y), since $x^2 + y^2 + 1 \geq 1$.

2. Yes. On the left-hand side we get $(2x + y)\, 2y \cos(xy^2) \sin(xy^2)$ and on the right-hand side we get $(2x + y)\, 2y \cos(xy^2) \left|\sin xy^2\right|$, so these are equal for $\sin(xy^2) \geq 0$.

3. Yes. There are many examples of such functions. One which works for all x and y is $f(x, y) = e^x + e^{-y}$, which has $f_x = e^x$ and $f_y = -e^{-y}$. A good strategy is to write $f(x, y) = g(x) + h(y)$, where $g'(x) > 0$, $h'(y) < 0$.

4.



5. $f(x,y) = x^2 + xy + y^2$ on the disk $\{(x,y) \mid x^2 + y^2 \le 9\}$.
$\nabla f(x,y) = \langle 2x + y, 2y + x \rangle = \langle 0,0 \rangle \Leftrightarrow y = -2x$ and $x = -2y \Leftrightarrow (x,y) = (0,0)$. So the minimum value on the interior of the disk is $f(0,0) = 0$.
Using Lagrange multipliers for the boundary, we solve $\nabla f = \lambda \nabla g$ where $g(x,y) = x^2 + y^2 = 9$. So $2x + y = \lambda 2x \Rightarrow \lambda = 1 + y/(2x)$ and $2y + x = \lambda 2y \Rightarrow \lambda = 1 + x/2y \Rightarrow x^2 = y^2$. But $x^2 + y^2 = 9$, so $2x^2 = 9 \Rightarrow x = \pm\frac{3}{\sqrt{2}}$ and $y = \pm\frac{3}{\sqrt{2}}$. Thus the maximum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$ and the minimum value on the boundary is $f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2}$.
The absolute minimum value is $f(0,0) = 0$ and the absolute maximum value is $f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{27}{2}$.

6. (a) Let $g(x,y,z) = \dfrac{x^2}{4} + \dfrac{y^2}{4} + 2z^2$, so $\nabla g = \left\langle \frac{x}{2}, \frac{y}{2}, 4z \right\rangle$ and $\nabla g\left(\sqrt{2}, \sqrt{2}, 0\right) = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$ which is normal to the surface. So the tangent plane satisfies $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = k$ and goes through $\left(\sqrt{2}, \sqrt{2}, 0\right)$. Thus $k = 1$ and the tangent plane is $\dfrac{x}{\sqrt{2}} + \dfrac{y}{\sqrt{2}} = 1$.

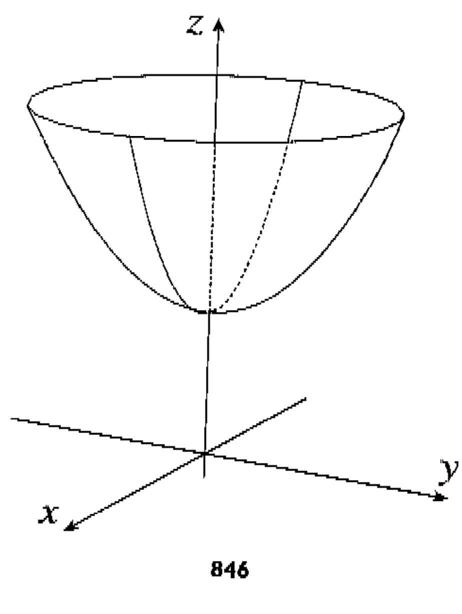
(b) Since this is a maximum value of z, the tangent plane is horizontal, that is, $z = \frac{1}{\sqrt{2}}$. Analytically, $\nabla g\left(0, 0, \frac{1}{\sqrt{2}}\right) = \left\langle 0, 0, 2\sqrt{2} \right\rangle$, so the tangent plane is $2\sqrt{2}z = 2$ or $z = \frac{1}{\sqrt{2}}$.

7. Ellipsoid for $k = -1$, single point $(0,0,0)$ for $k = 1$, no surface for $k = 2$.

8. (a) $F(x,y,z) = \dfrac{z}{1 + x^2 + y^2}$

(b) $z = 0$ is the only place where $F(x,y,z) = 0$. So there is no energy on the xy-plane.

(c) $F(x,y,z) = 1$ gives $1 = \dfrac{z}{1 + x^2 + y^2}$ or $z = 1 + x^2 + y^2$, a circular paraboloid.



9. $f(x, y) = \dfrac{x + y}{|x| + |y|}$

(a) (i) $f(1, 1) = 1$

(ii) $f(1, -1) = 0$

(iii) $f(-1, 1) = 0$

(iv) $f(-1, -1) = -1$

(b) No, the function does not have a limit at $(0, 0)$, since if $y = -x$, then $f(x, -x) = 0$ and if $y = x$, $f(x, x) = \dfrac{x}{|x|} = \pm 1$.

10. $f(x, y) = \begin{cases} \dfrac{2x^2 + 3y^2}{x - y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

(a) $f_x(0, 0) = \lim_{h \to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \dfrac{\frac{2h^2}{h} - 0}{h} = \lim_{h \to 0} \dfrac{2h^2}{h^2} = \lim_{h \to 0} 2 = 2$

(b) $f(0, y) = \dfrac{3y^2}{-y} = 3y = g(y)$. Then $f_y(0, 0) = g'(0) = -3$.

11. A counterexample is $f(x, y) = x^2 + y^2$. For this function $f_x(0, 0) = f_y(0, 0) = 0$; $f(0, 0) = 0$ and $f(x, y) \neq 0$ for $(x, y) \neq (0, 0)$.

12. $x^2 + y^2 + z^2 = 9$

(a) The tangent plane at $(3, 0, 0)$ is $x = 3$.

(b) Let $g(x, y, z) = x^2 + y^2 + z^2$. Then $\nabla g = \langle 2x, 2y, 2z \rangle$ and $\nabla g(2, 2, 1) = \langle 4, 4, 2 \rangle$, which is normal to the surface. So the tangent plane is $4x + 4y + 2z = k$ and goes through $(2, 2, 1)$, so $k = 18$, and the tangent plane is $2x + 2y + z = 9$.

13. $f(x, y) = e^{x-y}$, $f_x(x, y) = e^{x-y}$, $f_y(x, y) = -e^{x-y}$.
$L(x, y) = f(\ln 2, \ln 2) + f_x(\ln 2, \ln 2)(x - \ln 2) + f_y(\ln 2, \ln 2)(y - \ln 2)$. So the linear approximation is $f(\ln 2 + 0.1, \ln 2 + 0.04) \approx L(\ln 2 + 0.1, \ln 2 + 0.04) = 1 + 1(0.1) - 1(0.04) = 1.06$.

14. (a) $yf_x = y\left[g'(x^2 + y^2)\, 2x\right] = 2xyg'(x^2 + y^2)$, $xf_y = x\left[g'(x^2 + y^2)\, 2y\right] = 2xyg'(x^2 + y^2)$.

(b) The maximal increase is in the direction of $\mathbf{u} = \langle 2g'(2), 2g'(2) \rangle$, which is the same as that of $\mathbf{w} = \langle 1, 1 \rangle$.

15. (a) True; the partials are continuous.

(b) True (in fact the plane is $z = 0$).

(c) False; if they were continuous, then we would have $f_{xy}(0, 0) = f_{yx}(0, 0)$.

(d) False; the linear approximation is $L(x, y) = 0$.

16. $\mathbf{u} = \langle 1, 0 \rangle$, $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, $D_\mathbf{u}\,(f\,(a,b)) = 3$ and $D_\mathbf{v}\,(f\,(a,b)) = \sqrt{2}$

(a) $\nabla f\,(a,b) = \langle f_1, f_2 \rangle$ and $\langle f_1, f_2 \rangle \cdot \mathbf{u} = 3 \;\Rightarrow\; f_1 = 3$. $\langle f_1, f_2 \rangle \cdot \mathbf{v} = \sqrt{2} \;\Rightarrow\; \dfrac{f_1}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2}$ $\Rightarrow\; \dfrac{3}{\sqrt{2}} + \dfrac{f_2}{\sqrt{2}} = \sqrt{2} \;\Rightarrow\; 3 + f_2 = 2 \;\Rightarrow\; f_2 = -1$. So $\nabla f\,(a,b) = \langle 3, -1 \rangle$.

(b) $D_\mathbf{w}\,(f\,(a,b))$ is maximized when $\mathbf{w}$ is in the direction of $\langle 3, -1 \rangle$. So $\mathbf{w} = \left\langle \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle$ and since $\mathbf{w} = \frac{4}{\sqrt{10}}\mathbf{u} - \frac{1}{\sqrt{5}}\mathbf{v}$, $D_\mathbf{w}\,(f\,(a,b)) = \frac{4}{\sqrt{10}} D_\mathbf{u}\,(f\,(a,b)) - \frac{1}{\sqrt{5}} D_\mathbf{v}\,(f\,(a,b)) = \frac{4}{\sqrt{10}} \cdot 3 - \frac{1}{\sqrt{5}} \cdot \sqrt{2} = \sqrt{10}$

(c) $D_\mathbf{w}\,(f\,(a,b)) = 0$ if $\mathbf{w} \cdot \langle 3, -1 \rangle = 0$, so $3w_1 - w_2 = 0$ and $w_1^2 + w_2^2 = 1$ gives $w_1^2 + 9w_1^2 = 1$, $w_1 = \frac{1}{\sqrt{10}}$ and $w_2 = \frac{3}{\sqrt{10}}$, so $\mathbf{w} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$.

17. Since f is a function which is constant on circles $x^2 + y^2 = R$ and since f is decreasing as the radius of the circle increases, then the maximum is $f\,(0,0) = 1$ and the minimum is $f\,(4,3) = e^{-25}$.

18. Let $d^2 = x^2 + y^2 + z^2$ and minimize d^2 subject to the constraint $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, $x, y, z > 0$. The method of Lagrange multipliers gives the point $(3, 3, 3)$.