CONCEPT CHECK

1. (a) A double Riemann sum of f is $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$, where ΔA is the area of each subrectangle and $\left(x_{ij}^*, y_{ij}^*\right)$ is a sample point in each subrectangle. If $f(x,y) \geq 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

   (b) $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$

   (c) If $f(x,y) \geq 0$, $\iint_R f(x,y)\,dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x,y)$. If f takes on both positive and negative values, $\iint_R f(x,y)\,dA$ is the difference of the volume above R but below the surface $z = f(x,y)$ and the volume below R but above the surface $z = f(x,y)$.

   (d) We usually evaluate $\iint_R f(x,y)\,dA$ as an iterated integral according to Fubini's Theorem (see Theorem 16.2.4 [ET 15.2.4]).

   (e) The Midpoint Rule for Double Integrals says that we approximate the double integral $\iint_R f(x,y)\,dA$ by the double Riemann sum $\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x}_i, \overline{y}_j\right)\Delta A$ where the sample points $\left(\overline{x}_i, \overline{y}_j\right)$ are the centers of the subrectangles.

   (f) $f_{\text{ave}} = \dfrac{1}{A(R)}\displaystyle\iint_R f(x,y)\,dA$ where $A(R)$ is the area of R.

2. (a) See (1) and (2) and the accompanying discussion in Section 16.3 [ET 15.3].

   (b) See (3) and the accompanying discussion in Section 16.3 [ET 15.3].

   (c) See (5) and the preceding discussion in Section 16.3 [ET 15.3].

   (d) See (6)–(11) in Section 16.3 [ET 15.3].

3. We may want to change from rectangular to polar coordinates in a double integral if the region R of integration is more easily described in polar coordinates. To accomplish this, we use $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$ where R is given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$.

4. (a) $m = \iint_D \rho(x, y)\, dA$

(b) $M_x = \iint_D y\rho(x, y)\, dA$, $M_y = \iint_D x\rho(x, y)\, dA$

(c) The center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$.

(d) $I_x = \iint_D y^2 \rho(x, y)\, dA$, $I_y = \iint_D x^2 \rho(x, y)\, dA$, $I_0 = \iint_D (x^2 + y^2)\rho(x, y)\, dA$

5. (a) $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$

(b) $f(x, y) \ge 0$ and $\iint_{\mathbb{R}^2} f(x, y)\, dA = 1$.

(c) The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} x f(x, y)\, dA$; the expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} y f(x, y)\, dA$.

6. (a) $\iiint_B f(x, y, z)\, dV = \lim\limits_{l, m, n \to \infty} \sum\limits_{i=1}^{l} \sum\limits_{j=1}^{m} \sum\limits_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V$

(b) We usually evaluate $\iiint_B f(x, y, z)\, dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals (see Theorem 16.6.4 [ET 15.6.4]).

(c) See the paragraph following Example 16.6.1 [ET 15.6.1].

(d) See (5) and (6) and the accompanying discussion in Section 16.6 [ET 15.6].

(e) See (10) and the accompanying discussion in Section 16.6 [ET 15.6].

(f) See (11) and the preceding discussion in Section 16.6 [ET 15.6].

7. (a) $m = \iiint_E \rho(x, y, z)\, dV$

(b) $M_{yz} = \iiint_E x\rho(x, y, z)\, dV$, $M_{xz} = \iiint_E y\rho(x, y, z)\, dV$, $M_{xy} = \iiint_E z\rho(x, y, z)\, dV$.

(c) The center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$.

(d) $I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\, dV$, $I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\, dV$, $I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\, dV$.

8. (a) See Formula 16.7.4 [ET 15.7.4] and the accompanying discussion.

(b) See Formula 16.8.3 [ET 15.8.3] and the accompanying discussion.

(c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

9. (a) $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 16.9 [ET 15.9].

(c) See (13) and the accompanying discussion in Section 16.9 [ET 15.9].

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. False. $\int_0^1 \int_0^x \sqrt{x+y^2}\,dy\,dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1 \int_y^1 \sqrt{x+y^2}\,dx\,dy$.

3. True by Equation 16.2.5 [ET 15.2.5].

4. $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^1 e^{y^2} \sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2} \sin y$ is an odd function. Therefore the statement is true.

5. True: $\iint_D \sqrt{4-x^2-y^2}\,dA =$ the volume under the surface $x^2+y^2+z^2=4$ and above the xy-plane
$= \frac{1}{2}$ (the volume of the sphere $x^2+y^2+z^2=4$) $= \frac{1}{2} \cdot \frac{4}{3}\pi(2)^3 = \frac{16}{3}\pi$

6. This statement is true because in the given region, $(x^2+\sqrt{y})\sin(x^2y^2) \le (1+2)(1) = 3$, so
$\int_1^4 \int_0^1 (x^2+\sqrt{y})\sin(x^2y^2)\,dx\,dy \le \int_1^4 \int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

7. The volume enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z=2$ is, in cylindrical coordinates,
$V = \int_0^{2\pi} \int_0^2 \int_r^2 r\,dz\,dr\,d\theta \neq \int_0^{2\pi} \int_0^2 \int_r^2 dz\,dr\,d\theta$, so the assertion is false.

8. True. The moment of inertia about the z-axis of a solid E with constant density k is
$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta$.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m=n=3$ and the sample points the upper right corners of each square, so

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A$$
$$= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2)+f(2,3)+f(3,1)+f(3,2)+f(3,3)]$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m=n=3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(\overline{x}_i,\overline{y}_j)\,\Delta A$$
$$= \Delta A\,[f(0.5,0.5)+(0.5,1.5)+(0.5,2.5)+(1.5,0.5)+f(1.5,1.5)$$
$$+f(1.5,2.5)+(2.5,0.5)+f(2.5,1.5)+f(2.5,2.5)]$$
$$\approx 1[1.2+2.5+5.0+3.2+4.5+7.1+5.2+6.5+9.0] = 44.2$$

3. $\int_1^2 \int_0^2 (y + 2xe^y)\,dx\,dy = \int_1^2 \left[xy + x^2 e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y + 4e^y)\,dy = \left[y^2 + 4e^y\right]_1^2$

$= 4 + 4e^2 - 1 - 4e = 4e^2 - 4e + 3$

4. $\int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1)\,dy = \left[e^y - y\right]_0^1 = e - 2$

5. $\int_0^1 \int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1 \int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x} - x^4)\,dx = \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ [integrate by parts in the first term]

$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

7. $\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi \int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi \int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$

$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

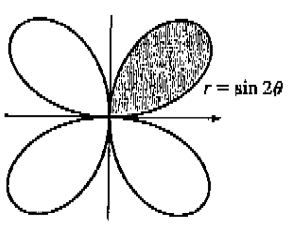
8. $\int_0^1 \int_0^y \int_x^1 6xyz\,dz\,dx\,dy = \int_0^1 \int_0^y \left[3xyz^2\right]_{z=x}^{z=1} dx\,dy = \int_0^1 \int_0^y (3xy - 3x^3y)\,dx\,dy$

$= \int_0^1 \left[\frac{3}{2}x^2y - \frac{3}{4}x^4y\right]_{x=0}^{x=y} dy = \int_0^1 \left(\frac{3}{2}y^3 - \frac{3}{4}y^5\right) dy = \left[\frac{3}{8}y^4 - \frac{1}{8}y^6\right]_0^1 = \frac{1}{4}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4,\ 0 \le \theta \le \pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^\pi \int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid y - 4 \le x \le 4 - y,\ 0 \le y \le 4\}$ and $\iint_R f(x,y)\,dA = \int_0^4 \int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

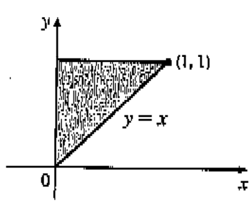
11.



The region whose area is given by $\int_0^{\pi/2} \int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\left\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le \sin 2\theta\right\}$, which is the region contained in the loop in the first quadrant of the four-leaved rose $r = \sin 2\theta$.

12. The solid is $\left\{(\rho,\theta,\phi) \mid 1 \le \rho \le 2,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}\right\}$ which is the region in the first octant on or between the two spheres $\rho = 1$ and $\rho = 2$.

13.

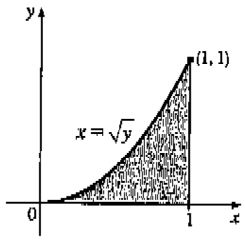


$\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx = \int_0^1 \int_0^y \cos(y^2)\,dx\,dy$

$= \int_0^1 \cos(y^2)\left[x\right]_{x=0}^{x=y} dy = \int_0^1 y\cos(y^2)\,dy$

$= \left[\frac{1}{2}\sin(y^2)\right]_0^1 = \frac{1}{2}\sin 1$

14.



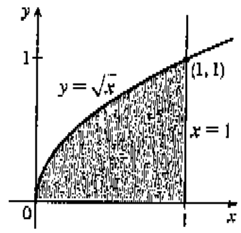
$$\int_0^1\int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy = \int_0^1\int_0^{x^2}\frac{ye^{x^2}}{x^3}\,dy\,dx = \int_0^1 \frac{e^{x^2}}{x^3}\left[\tfrac12 y^2\right]_{y=0}^{y=x^2}dx$$
$$= \int_0^1 \tfrac12 xe^{x^2}\,dx = \tfrac14 e^{x^2}\Big]_0^1 = \tfrac14(e-1)$$

15. $\iint_R ye^{xy}\,dA = \int_0^3\int_0^2 ye^{xy}\,dx\,dy = \int_0^3\left[e^{xy}\right]_{x=0}^{x=2}dy = \int_0^3(e^{2y}-1)\,dy = \left[\tfrac12 e^{2y}-y\right]_0^3 = \tfrac12 e^6 - 3 - \tfrac12 = \tfrac12 e^6 - \tfrac72$

16. $\iint_D xy\,dA = \int_0^1\int_{y^2}^{y+2} xy\,dx\,dy = \int_0^1 y\left[\tfrac12 x^2\right]_{x=y^2}^{x=y+2}dy = \tfrac12\int_0^1 y((y+2)^2-y^4)\,dy$

$= \tfrac12\int_0^1(y^3+4y^2+4y-y^5)\,dy = \tfrac12\left[\tfrac14 y^4 + \tfrac43 y^3 + 2y^2 - \tfrac16 y^6\right]_0^1 = \tfrac{41}{24}$

17.

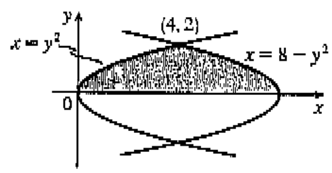


$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1\int_0^{\sqrt{x}}\frac{y}{1+x^2}\,dy\,dx = \int_0^1\frac{1}{1+x^2}\left[\tfrac12 y^2\right]_{y=0}^{y=\sqrt{x}}dx$$
$$= \tfrac12\int_0^1\frac{x}{1+x^2}\,dx = \left[\tfrac14\ln(1+x^2)\right]_0^1 = \tfrac14\ln 2$$

18. $\iint_D \frac{1}{1+x^2}\,dA = \int_0^1\int_x^1\frac{1}{1+x^2}\,dy\,dx = \int_0^1\frac{1}{1+x^2}\left[y\right]_{y=x}^{y=1}dx = \int_0^1\frac{1-x}{1+x^2}\,dx = \int_0^1\left(\frac{1}{1+x^2}-\frac{x}{1+x^2}\right)dx$

$= \left[\tan^{-1}x - \tfrac12\ln(1+x^2)\right]_0^1 = \tan^{-1}1 - \tfrac12\ln 2 - \left(\tan^{-1}0 - \tfrac12\ln 1\right) = \tfrac{\pi}{4} - \tfrac12\ln 2$

19.

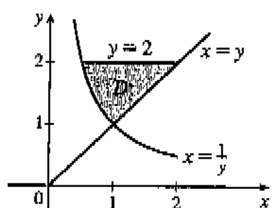


$\iint_D y\,dA = \int_0^2\int_{y^2}^{8-y^2} y\,dx\,dy$

$= \int_0^2 y\left[x\right]_{x=y^2}^{x=8-y^2}dy = \int_0^2 y(8-y^2-y^2)\,dy$

$= \int_0^2(8y-2y^3)\,dy = \left[4y^2-\tfrac12 y^4\right]_0^2 = 8$

20.



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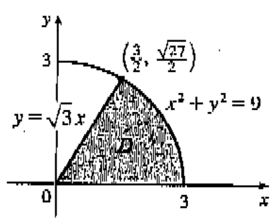
•
15.5 24 33 •
•
15.6 24, 25, 34 50 •
•
15.7 12, 27 •
•
15.8 1, 4, 10, 17, 21,

$\ldots y = \int_1^2 y\left(y-\frac1y\right)dy$

$= \left[\tfrac13 y^3 - y\right]_1^2$

$\ldots - 1) = \tfrac43$

21.



$\iint_D \ldots = \int_0^{\pi/3}\int_0^3 (r^2)^{3/2}\,r\,dr\,d\theta$

$= \int_0^{\pi/3}d\theta\int_0^3 r^4\,dr = \left[\theta\right]_0^{\pi/3}\left[\frac15 r^5\right]_0^3$

$= \frac{\pi}{3}\,\frac{3^5}{5} = \frac{81\pi}{5}$

22. $\iint_D x\,dA = \int_0^{\pi/2}\int_1^{\sqrt2}(r\cos\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\,d\theta\int_1^{\sqrt2}r^2\,dr = \left[\sin\theta\right]_0^{\pi/2}\left[\tfrac13 r^3\right]_1^{\sqrt2}$

$= 1\cdot\tfrac13(2^{3/2}-1) = \tfrac13(2^{3/2}-1)$

23. $\iiint_E xy\,dV = \int_0^3 \int_0^x \int_0^{x+y} xy\,dz\,dy\,dx = \int_0^3 \int_0^x xy\left[z\right]_{z=0}^{z=x+y} dy\,dx = \int_0^3 \int_0^x xy(x+y)\,dy\,dx$

$= \int_0^3 \int_0^x (x^2y + xy^2)\,dy\,dx = \int_0^3 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_{y=0}^{y=x} dx = \int_0^3 \left(\frac{1}{2}x^4 + \frac{1}{3}x^4\right) dx$

$= \frac{5}{6}\int_0^3 x^4\,dx = \left[\frac{1}{6}x^5\right]_0^3 = \frac{81}{2} = 40.5$

24. $\iiint_T xy\,dV = \int_0^{1/3} \int_0^{1-3x} \int_0^{1-3x-y} xy\,dz\,dy\,dx = \int_0^{1/3} \int_0^{1-3x} xy(1-3x-y)\,dy\,dx$

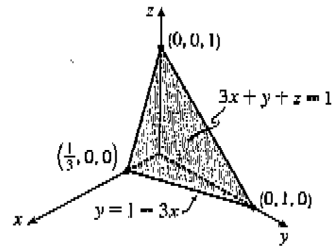
$= \int_0^{1/3} \int_0^{1-3x} (xy - 3x^2y - xy^2)\,dy\,dx$

$= \int_0^{1/3} \left[\frac{1}{2}xy^2 - \frac{3}{2}x^2y^2 - \frac{1}{3}xy^3\right]_{y=0}^{y=1-3x} dx$

$= \int_0^{1/3} \left[\frac{1}{2}x(1-3x)^2 - \frac{3}{2}x^2(1-3x)^2 - \frac{1}{3}x(1-3x)^3\right] dx$

$= \int_0^{1/3} \left(\frac{1}{6}x - \frac{3}{2}x^2 + \frac{9}{2}x^3 - \frac{9}{2}x^4\right) dx$

$= \frac{1}{12}x^2 - \frac{1}{2}x^3 + \frac{9}{8}x^4 - \frac{9}{10}x^5\Big]_0^{1/3} = \frac{1}{1080}$



25. $\iiint_E y^2z^2\,dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2(1-y^2-z^2)\,dz\,dy$

$= \int_0^{2\pi} \int_0^1 (r^2\cos^2\theta)(r^2\sin^2\theta)(1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{4}\sin^2 2\theta(r^5 - r^7)\,dr\,d\theta$

$= \int_0^{2\pi} \frac{1}{8}(1-\cos 4\theta)\left[\frac{1}{6}r^6 - \frac{1}{8}r^8\right]_{r=0}^{r=1} d\theta = \frac{1}{192}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{2\pi}{192} = \frac{\pi}{96}$

26. $\iiint_E z\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{2-y} z\,dx\,dz\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} (2-y)z\,dz\,dy = \int_0^1 \frac{1}{2}(2-y)(1-y^2)\,dy$

$= \int_0^1 \frac{1}{2}(2 - y - 2y^2 + y^3)\,dy = \frac{13}{24}$

27. $\iiint_E yz\,dV = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_0^y yz\,dz\,dy\,dx = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \frac{1}{2}y^3\,dy\,dx = \int_0^\pi \int_0^2 \frac{1}{2}r^3(\sin^3\theta)\,r\,dr\,d\theta$

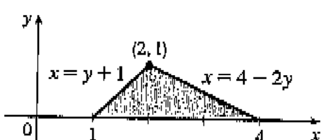
$= \frac{16}{5}\int_0^\pi \sin^3\theta\,d\theta = \frac{16}{5}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^\pi = \frac{64}{15}$

28. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho^3\cos^3\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos^3\phi\sin\phi\,d\phi \int_0^1 \rho^6\,d\rho = 2\pi\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}\left(\frac{1}{7}\right) = \frac{\pi}{14}$

29. $V = \int_0^2 \int_1^4 (x^2 + 4y^2)\,dy\,dx = \int_0^2 \left[x^2y + \frac{4}{3}y^3\right]_{y=1}^{y=4} dx = \int_0^2 (3x^2 + 84)\,dx = 176$

30.

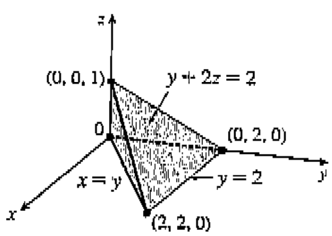


$V = \int_0^1 \int_{y+1}^{4-2y} \int_0^{x^2y} dz\,dx\,dy = \int_0^1 \int_{y+1}^{4-2y} x^2y\,dx\,dy$

$= \int_0^1 \frac{1}{3}\left[(4-2y)^3y - (y+1)^3y\right] dy$

$= \int_0^1 3(-y^4 + 5y^3 - 11y^2 + 7y)\,dy = 3\left(-\frac{1}{5} + \frac{5}{4} - \frac{11}{3} + \frac{7}{2}\right) = \frac{53}{20}$

31.



$V = \int_0^2 \int_0^y \int_0^{(2-y)/2} dz\,dx\,dy = \int_0^2 \int_0^y \left(1 - \frac{1}{2}y\right) dx\,dy$

$= \int_0^2 \left(y - \frac{1}{2}y^2\right) dy = \frac{2}{3}$

32. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r - r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi}\left[6 - \frac{8}{3}\sin\theta\right]d\theta = 6\theta\big]_0^{2\pi} + 0 = 12\pi$

33. Using the wedge above the plane $z = 0$ and below the plane $z = mx$ and noting that we have the same volume for $m < 0$ as for $m > 0$ (so use $m > 0$), we have

$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3}\frac{1}{2}m(a^2 - 9y^2)\,dy = m\left[a^2y - 3y^3\right]_0^{a/3} = m\left(\frac{1}{3}a^3 - \frac{1}{9}a^3\right) = \frac{2}{9}ma^3.$

34. The paraboloid and the half-cone intersect when $x^2 + y^2 = \sqrt{x^2+y^2}$, that is when $x^2 + y^2 = 1$ or 0. So

$V = \iint_{x^2+y^2\le 1}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^2 - r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{3} - \frac{1}{4}\right)d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6}.$

35. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 (y - y^3)\,dy = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac{1}{2}y(1-y^2)^2\,dy = -\frac{1}{12}(1-y^2)^3\big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 (y^2 - y^4)\,dy = \frac{2}{15}$. Hence $(\overline{x},\overline{y}) = \left(\frac{1}{3}, \frac{8}{15}\right)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 (y^3 - y^5)\,dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac{1}{3}y(1-y^2)^3\,dy = -\frac{1}{24}(1-y^2)^4\big]_0^1 = \frac{1}{24}$,

$I_0 = I_x + I_y = \frac{1}{8}$, $\overline{\overline{y}}^2 = \frac{1/12}{1/4} = \frac{1}{3} \Rightarrow \overline{\overline{y}} = \frac{1}{\sqrt{3}}$, and $\overline{\overline{x}}^2 = \frac{1/24}{1/4} = \frac{1}{6} \Rightarrow \overline{\overline{x}} = \frac{1}{\sqrt{6}}$.

36. (a) $m = \frac{1}{4}\pi K a^2$ where K is constant,

$M_y = \iint_{x^2+y^2\le a^2} Kx\,dA = K\int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{1}{3}Ka^3\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}a^3K$, and

$M_x = K\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{1}{3}a^3K$ [by symmetry $M_y = M_x$].

Hence the centroid is $(\overline{x},\overline{y}) = \left(\frac{4}{3\pi}a, \frac{4}{3\pi}a\right)$.

(b) $m = \int_0^{\pi/2}\int_0^a r^4\cos\theta\sin^2\theta\,dr\,d\theta = \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}\left(\frac{1}{5}a^5\right) = \frac{1}{15}a^5$,

$M_y = \int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{96}\pi a^6$, and

$M_x = \int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta = \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{24}a^6$. Hence $(\overline{x},\overline{y}) = \left(\frac{5}{32}\pi a, \frac{5}{8}a\right)$.

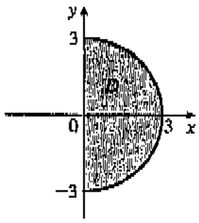
37. The equation of the cone with the suggested orientation is $(h - z) = \frac{h}{a}\sqrt{x^2+y^2}$, $0 \le z \le h$. Then $V = \frac{1}{3}\pi a^2 h$ is the volume of one frustum of a cone; by symmetry $M_{yz} = M_{xz} = 0$; and

$$M_{xy} = \iint_{x^2+y^2\le a^2}\int_0^{h-(h/a)\sqrt{x^2+y^2}} z\,dz\,dA = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} rz\,dz\,dr\,d\theta = \pi\int_0^a r\frac{h^2}{a^2}(a-r)^2\,dr$$

$$= \frac{\pi h^2}{a^2}\int_0^a (a^2r - 2ar^2 + r^3)\,dr = \frac{\pi h^2}{a^2}\left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{\pi h^2 a^2}{12}$$

Hence the centroid is $(\overline{x},\overline{y},\overline{z}) = \left(0, 0, \frac{1}{4}h\right)$.

38. $$I_z = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} r^3\,dz\,dr\,d\theta = 2\pi\int_0^a \frac{h}{a}(ar^3 - r^4)\,dr = \frac{2\pi h}{a}\left(\frac{a^5}{4} - \frac{a^5}{5}\right) = \frac{\pi a^4 h}{10}$$

39. 

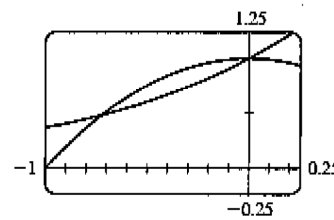
$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dy\, dx = \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x(x^2+y^2)\, dy\, dx$$
$$= \int_{-\pi/2}^{\pi/2} \int_0^3 (r\cos\theta)(r^2)\, r\, dr\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta \int_0^3 r^4\, dr$$
$$= \left[\sin\theta\right]_{-\pi/2}^{\pi/2} \left[\tfrac{1}{5}r^5\right]_0^3 = 2\cdot\tfrac{1}{5}(243) = \tfrac{486}{5} = 97.2$$

40. The region of integration is the solid hemisphere $x^2+y^2+z^2 \le 4$, $x \ge 0$.

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$
$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^2 (\rho\sin\phi\sin\theta)^2\left(\sqrt{\rho^2}\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\theta\, d\theta \int_0^{\pi}\sin^3\phi\, d\phi \int_0^2 \rho^5\, d\rho$$
$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^2 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{32}{3}\right) = \tfrac{64}{9}\pi$$

41. From the graph, it appears that $1 - x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1 - x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is

$$\iint_D y^2\, dA \approx \int_{-0.71}^0 \int_{e^x}^{1-x^2} y^2\, dy\, dx$$
$$= \tfrac{1}{3}\int_{-0.71}^0 \left[(1-x^2)^3 - e^{3x}\right] dx$$
$$= \tfrac{1}{3}\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{1}{3}e^{3x}\right]_{-0.71}^0 \approx 0.0512$$



42. Let the tetrahedron be called T. The front face of T is given by the plane $x + \frac{1}{2}y + \frac{1}{3}z = 1$, or $z = 3 - 3x - \frac{3}{2}y$, which intersects the xy-plane in the line $y = 2 - 2x$. So the total mass is

$m = \iiint_T \rho(x,y,z)\, dV = \int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2}(x^2+y^2+z^2)\, dz\, dy\, dx = \frac{7}{5}$. The center of mass is

$(\overline{x},\overline{y},\overline{z}) = \left(m^{-1}\iiint_T x\rho(x,y,z)\, dV, m^{-1}\iiint_T y\rho(x,y,z)\, dV, m^{-1}\iiint_T z\rho(x,y,z)\, dV\right) = \left(\frac{4}{21}, \frac{11}{21}, \frac{8}{7}\right)$.

43. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x,y)\, dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,3]\times[0,2]$, we can say

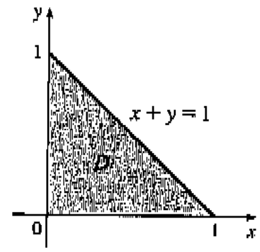
$$\iint_{\mathbb{R}^2} f(x,y)\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = \int_0^3\int_0^2 C(x+y)\, dy\, dx$$
$$= C\int_0^3 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=2} dx = C\int_0^3 (2x+2)\, dx = C\left[x^2+2x\right]_0^3 = 15C$$

Then $15C = 1 \;\Rightarrow\; C = \frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^2\int_1^{\infty} f(x,y)\, dy\, dx = \int_0^2\int_1^2 \frac{1}{15}(x,y)\, dy\, dx = \frac{1}{15}\int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2} dx$

$$= \tfrac{1}{15}\int_0^2 \left(x + \tfrac{3}{2}\right) dx = \tfrac{1}{15}\left[\tfrac{1}{2}x^2 + \tfrac{3}{2}x\right]_0^2 = \tfrac{1}{3}$$

(c) $P(X+Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X+Y\le 1) = \iint_D f(x,y)\,dA = \int_0^1\int_0^{1-x} \tfrac{1}{15}(x+y)\,dy\,dx$$
$$= \tfrac{1}{15}\int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$
$$= \tfrac{1}{15}\int_0^1 \left[x(1-x) + \tfrac{1}{2}(1-x)^2\right] dx$$
$$= \tfrac{1}{30}\int_0^1 (1-x^2)\,dx = \tfrac{1}{30}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{45}$$

44. Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800}e^{-t/800} & \text{if } t \ge 0 \end{cases}$$

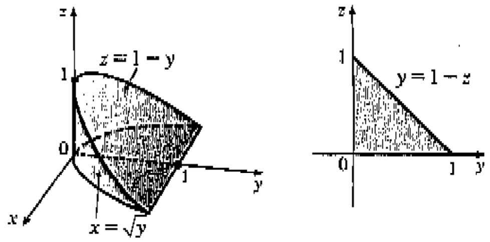
If X, Y, and Z are the lifetimes of the individual bulbs, then X, Y, and Z are independent, so the joint density function is the product of the individual density functions:

$$f(x,y,z) = \begin{cases} \frac{1}{800^3}e^{-(x+y+z)/800} & \text{if } x\ge 0, y \ge 0, z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P(X+Y+Z \le 1000)$, or equivalently $P((X,Y,Z)\in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x+y+z=1000$. The plane $x+y+z=1000$ meets the xy-plane in the line $x+y=1000$, so we have

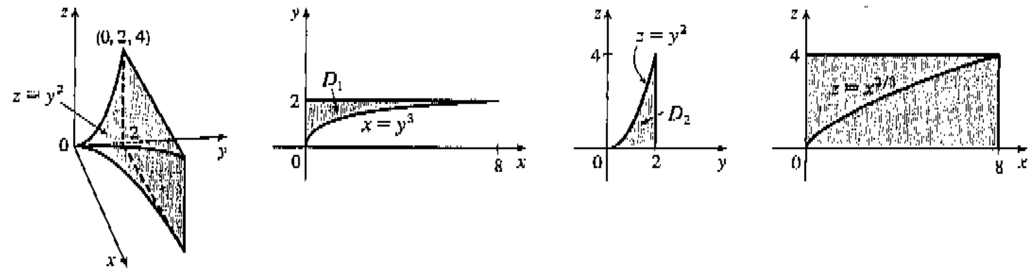
$$P(X+Y+Z\le 1000) = \iiint_E f(x,y,z)\,dV = \int_0^{1000}\int_0^{1000-x}\int_0^{1000-x-y} \tfrac{1}{800^3}e^{-(x+y+z)/800}\,dz\,dy\,dx$$
$$= \tfrac{1}{800^3}\int_0^{1000}\int_0^{1000-x} -800\left[e^{-(x+y+z)/800}\right]_{z=0}^{z=1000-x-y} dy\,dx$$
$$= \tfrac{-1}{800^2}\int_0^{1000}\int_0^{1000-x}\left[e^{-5/4} - e^{-(x+y)/800}\right] dy\,dx$$
$$= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}y + 800e^{-(x+y)/800}\right]_{y=0}^{y=1000-x} dx$$
$$= \tfrac{-1}{800^2}\int_0^{1000}\left[e^{-5/4}(1800-x) - 800e^{-x/800}\right] dx$$
$$= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(1800-x)^2 + 800^2e^{-x/800}\right]_0^{1000}$$
$$= \tfrac{-1}{800^2}\left[-\tfrac{1}{2}e^{-5/4}(800)^2 + 800^2e^{-5/4} + \tfrac{1}{2}e^{-5/4}(1800)^2 - 800^2\right]$$
$$= 1 - \tfrac{97}{32}e^{-5/4} \approx 0.1315$$

45.



$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$

46.



$\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y^3, 0 \le z \le y^2\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$D_1 = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y^3\} = \{(x,y) \mid 0 \le x \le 8, \sqrt[3]{x} \le y \le 2\}$,

$D_2 = \{(y,z) \mid 0 \le z \le 4, \sqrt{z} \le y \le 2\} = \{(y,z) \mid 0 \le y \le 2, 0 \le z \le y^2\}$, $D_3 = \{(x,z) \mid 0 \le x \le 8, 0 \le z \le 4\}$.

Therefore we have

$$\begin{aligned}\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy &= \int_0^8\int_{\sqrt[3]{x}}^2\int_0^{y^2} f(x,y,z)\,dz\,dy\,dx = \int_0^4\int_{\sqrt{z}}^2\int_0^{y^3} f(x,y,z)\,dx\,dy\,dz \\ &= \int_0^2\int_0^{y^2}\int_0^{y^3} f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^8\int_0^{x^{2/3}}\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dz\,dx + \int_0^8\int_{x^{2/3}}^4\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^4\int_0^{z^{3/2}}\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dx\,dz + \int_0^4\int_{z^{3/2}}^8\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

47. Since $u = x - y$ and $v = x + y$, $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(v - u)$.

Thus $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$ and $\displaystyle\iint_R \frac{x-y}{x+y}\,dA = \int_2^4\int_{-2}^{0} \frac{u}{v}\left(\frac{1}{2}\right) du\,dv = -\int_2^4 \frac{dv}{v} = -\ln 2$.

48. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$, so

$$\begin{aligned} V &= \iiint_E dV = \int_0^1\int_0^{1-u}\int_0^{1-u-v} 8uvw\,dw\,dv\,du = \int_0^1\int_0^{1-u} 4uv(1-u-v)^2\,du \\ &= \int_0^1\int_0^{1-u}\left[4u(1-u)^2 v - 8u(1-u)v^2 + 4uv^3\right] dv\,du \\ &= \int_0^1 \left[2u(1-u)^4 - \tfrac{8}{3}u(1-u)^4 + u(1-u)^4\right] du = \int_0^1 \tfrac{1}{3}u(1-u)^4\,du \\ &= \int_0^1 \tfrac{1}{3}\left[(1-u)^4 - (1-u)^5\right] du = \tfrac{1}{3}\left[-\tfrac{1}{5}(1-u)^5 + \tfrac{1}{6}(1-u)^6\right]_0^1 = \tfrac{1}{3}\left(-\tfrac{1}{6} + \tfrac{1}{5}\right) = \tfrac{1}{90}\end{aligned}$$

49. Let $u = y - x$ and $v = y + x$ so $x = y - u = (v - x) - u \Rightarrow x = \frac{1}{2}(v - u)$ and $y = v - \frac{1}{2}(v - u) = \frac{1}{2}(v + u)$.

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| = \left|-\frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$. R is the image under this transformation of the square with vertices $(u, v) = (0, 0)$, $(-2, 0)$, $(0, 2)$, and $(-2, 2)$. So

$$\iint_R xy\,dA = \int_0^2 \int_{-2}^0 \frac{v^2 - u^2}{4}\left(\frac{1}{2}\right) du\,dv = \tfrac{1}{8}\int_0^2 \left[v^2 u - \tfrac{1}{3}u^3\right]_{u=-2}^{u=0} dv = \tfrac{1}{8}\int_0^2 \left(2v^2 - \tfrac{8}{3}\right) dv = \tfrac{1}{8}\left[\tfrac{2}{3}v^3 - \tfrac{8}{3}v\right]_0^2 = 0$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

50. By the Extreme Value Theorem (15.7.8 [ET 14.7.8]), f has an absolute minimum value m and an absolute maximum value M in D. Then by Property 16.3.11 [ET 15.3.11], $mA(D) \le \iint_D f(x, y)\,dA \le MA(D)$. Dividing through by the positive number $A(D)$, we get $m \le \dfrac{1}{A(D)}\displaystyle\iint_D f(x, y)\,dA \le M$. This says that the average value of f over D lies between m and M. But f is continuous on D and takes on the values m and M, and so by the Intermediate Value Theorem must take on all values between m and M. Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \dfrac{1}{A(D)}\displaystyle\iint_D f(x, y)\,dA$ or equivalently $\iint_D f(x, y)\,dA = f(x_0, y_0)\,A(D)$.

51. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for Double Integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \dfrac{1}{\pi r^2}\displaystyle\iint_{D_r} f(x, y)\,dA$. But $\lim\limits_{r\to 0^+}(x_r, y_r) = (a, b)$,

so $\lim\limits_{r\to 0^+}\dfrac{1}{\pi r^2}\displaystyle\iint_{D_r} f(x, y)\,dA = \lim\limits_{r\to 0^+} f(x_r, y_r) = f(a, b)$ by the continuity of f.

52. (a) $$\iint_D \frac{1}{(x^2 + y^2)^{n/2}}\,dA = \int_0^{2\pi}\int_r^R \frac{1}{(t^2)^{n/2}}\,t\,dt\,d\theta = 2\pi\int_r^R t^{1-n}\,dt$$

$$= \begin{cases} \left.\dfrac{2\pi}{2-n}t^{2-n}\right]_r^R = \dfrac{2\pi}{2-n}\left(R^{2-n} - r^{2-n}\right) & \text{if } n \ne 2 \\ 2\pi\ln(R/r) & \text{if } n = 2 \end{cases}$$

(b) The integral in part (a) has a limit as $r \to 0^+$ for all values of n such that $2 - n > 0 \Leftrightarrow n < 2$.

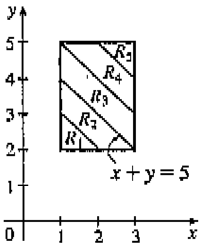
(c) $$\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}}\,dV = \int_r^R\int_0^\pi\int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}}\rho^2\sin\phi\,d\theta\,d\phi\,d\rho = 2\pi\int_r^R\int_0^\pi \rho^{2-n}\sin\phi\,d\phi\,d\rho$$

$$= \begin{cases} \left.\dfrac{4\pi}{3-n}\rho^{3-n}\right]_r^R = \dfrac{4\pi}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{if } n \ne 3 \\ 4\pi\ln(R/r) & \text{if } n = 3 \end{cases}$$

(d) As $r \to 0^+$, the above integral has a limit, provided that $3 - n > 0 \Leftrightarrow n < 3$.

PROBLEMS PLUS

1.



Let $R = \bigcup_{i=1}^{5} R_i$, where

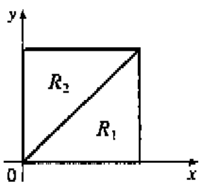
$R_i = \{(x,y) \mid x+y \geq i+2, x+y < i+3, 1 \leq x \leq 3, 2 \leq y \leq 5\}$.

$\iint_R [\![x+y]\!]\,dA = \sum_{i=1}^{5} \iint_{R_i} [\![x+y]\!]\,dA = \sum_{i=1}^{5} [\![x+y]\!] \iint_{R_i} dA$, since

$[\![x+y]\!] = \text{constant} = i+2$ for $(x,y) \in R_i$. Therefore

$$\begin{aligned}\iint_R [\![x+y]\!]\,dA &= \textstyle\sum_{i=1}^{5} (i+2)\,[A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\tfrac{1}{2}\right) + 4\left(\tfrac{3}{2}\right) + 5(2) + 6\left(\tfrac{3}{2}\right) + 7\left(\tfrac{1}{2}\right) = 30\end{aligned}$$

2.



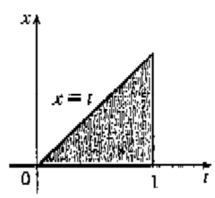
Let $R = \{(x,y) \mid 0 \leq x, y \leq 1\}$. For $x, y \in R$, $\max\{x^2, y^2\} = x^2$ if $x \geq y$, and $\max\{x^2, y^2\} = y^2$ if $x \leq y$. Therefore we divide R into two regions: $R = R_1 \cup R_2$, where $R_1 = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ and $R_2 = \{(x,y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$. Now $\max\{x^2, y^2\} = x^2$ for $(x,y) \in R_1$, and $\max\{x^2, y^2\} = y^2$ for $(x,y) \in R_2 \quad \Rightarrow$

$$\begin{aligned}\int_0^1\!\int_0^1 e^{\max\{x^2,y^2\}}\,dy\,dx &= \iint_R e^{\max\{x^2,y^2\}}\,dA = \iint_{R_1} e^{\max\{x^2,y^2\}}\,dA + \iint_{R_2} e^{\max\{x^2,y^2\}}\,dA \\ &= \int_0^1\!\int_0^x e^{x^2}\,dy\,dx + \int_0^1\!\int_0^y e^{y^2}\,dx\,dy = \int_0^1 xe^{x^2}\,dx + \int_0^1 ye^{y^2}\,dy = e^{x^2}\Big]_0^1 = e - 1\end{aligned}$$

3. $$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{1-0}\int_0^1 \left[\int_x^1 \cos(t^2)\,dt\right] dx$$

$$= \int_0^1\!\int_x^1 \cos(t^2)\,dt\,dx = \int_0^1\!\int_0^t \cos(t^2)\,dx\,dt \quad \text{[changing the order of integration]}$$

$$= \int_0^1 t\cos(t^2)\,dt = \tfrac{1}{2}\sin(t^2)\Big]_0^1 = \tfrac{1}{2}\sin 1$$



4. Let $u = \mathbf{a} \cdot \mathbf{r}$, $v = \mathbf{b} \cdot \mathbf{r}$, $w = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$. Under this change of variables, E corresponds to the rectangular box $0 \leq u \leq \alpha$, $0 \leq v \leq \beta$, $0 \leq w \leq \gamma$. So, by Formula 16.9.13 [ET 15.9.13],

$$\int_0^\gamma\!\int_0^\beta\!\int_0^\alpha uvw\,du\,dv\,dw = \iiint_E (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r}) \left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| dV. \text{ But}$$

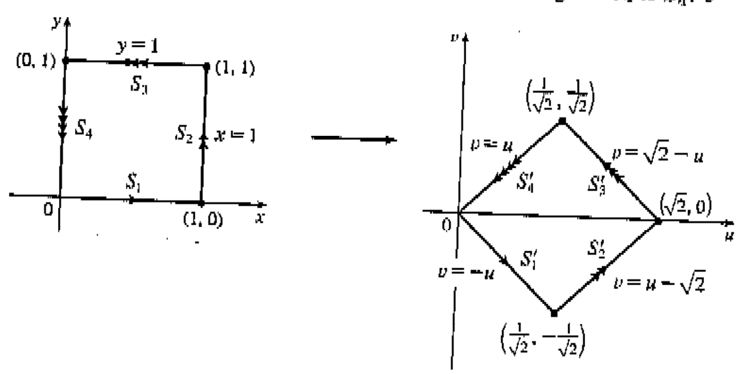
$$\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\| = |\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}| \quad \Rightarrow$$

$$\begin{aligned}\iiint_E (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r})\,dV &= \frac{1}{|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|}\int_0^\gamma\!\int_0^\beta\!\int_0^\alpha uvw\,du\,dv\,dw \\ &= \frac{1}{|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|}\left(\frac{\alpha^2}{2}\right)\left(\frac{\beta^2}{2}\right)\left(\frac{\gamma^2}{2}\right) = \frac{(\alpha\beta\gamma)^2}{8\,|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|}\end{aligned}$$

5. Since $|xy| < 1$, except at $(1,1)$, the formula for the sum of a geometric series gives $\frac{1}{1-xy} = \sum_{n=0}^{\infty} (xy)^n$, so

$$\int_0^1\int_0^1 \tfrac{1}{1-xy}\,dx\,dy = \int_0^1\int_0^1 \sum_{n=0}^{\infty}(xy)^n\,dx\,dy = \sum_{n=0}^{\infty}\int_0^1\int_0^1 (xy)^n\,dx\,dy = \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]$$
$$= \sum_{n=0}^{\infty}\tfrac{1}{n+1}\cdot\tfrac{1}{n+1} = \sum_{n=0}^{\infty}\tfrac{1}{(n+1)^2} = \tfrac{1}{1^2}+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\cdots = \sum_{n=1}^{\infty}\tfrac{1}{n^2}$$

6. Let $x = \frac{u-v}{\sqrt{2}}$ and $y = \frac{u+v}{\sqrt{2}}$. We know the region of integration in the xy-plane, so to find its image in the uv-plane we get u and v in terms of x and y, and then use the methods of Section 16.9 [ET 15.9]. $x + y = \frac{u-v}{\sqrt{2}} + \frac{u+v}{\sqrt{2}} = \sqrt{2}\,u$, so $u = \frac{x+y}{\sqrt{2}}$, and similarly $v = \frac{y-x}{\sqrt{2}}$. S_1 is given by $y = 0$, $0 \le x \le 1$, so from the equations derived above, the image of S_1 is S_1': $u = \frac{1}{\sqrt{2}}x$, $v = -\frac{1}{\sqrt{2}}x$, $0 \le x \le 1$, that is, $v = -u$, $0 \le u \le \frac{1}{\sqrt{2}}$. Similarly, the image of S_2 is S_2': $v = u - \sqrt{2}$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, the image of S_3 is S_3': $v = \sqrt{2} - u$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, and the image of S_4 is S_4': $v - u$, $0 \le u \le \frac{1}{\sqrt{2}}$.



The Jacobian of the transformation is $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$. From the diagram, we see that we must evaluate two integrals: one over the region $\left\{(u,v) \mid 0 \le u \le \frac{1}{\sqrt{2}},\ -u \le v \le u\right\}$ and the other over $\left\{(u,v) \mid \frac{1}{\sqrt{2}} \le u \le \sqrt{2},\ -\sqrt{2}+u \le v \le \sqrt{2}-u\right\}$. So

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]}$$
$$= \int_0^{\sqrt{2}/2}\int_{-u}^{u}\frac{2\,dv\,du}{2-u^2+v^2} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u}\frac{2\,dv\,du}{2-u^2+v^2}$$
$$= 2\left[\int_0^{\sqrt{2}/2}\frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-u}^{u} du + \int_{\sqrt{2}/2}^{\sqrt{2}}\frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-\sqrt{2}+u}^{\sqrt{2}-u} du\right]$$
$$= 4\left[\int_0^{\sqrt{2}/2}\frac{1}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}\,du + \int_{\sqrt{2}/2}^{\sqrt{2}}\frac{1}{\sqrt{2-u^2}}\arctan\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\,du\right]$$

Now let $u = \sqrt{2}\sin\theta$, so $du = \sqrt{2}\cos\theta\,d\theta$ and the limits change to 0 and $\frac{\pi}{6}$ (in the first integral) and $\frac{\pi}{6}$ and $\frac{\pi}{2}$ (in the second integral). Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \frac{1}{\sqrt{2-2\sin^2\theta}} \arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right) + \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2-2\sin^2\theta}} \arctan\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right)\right]$$

$$= 4\left[\int_0^{\pi/6} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta} \arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) d\theta + \int_{\pi/6}^{\pi/2} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta} \arctan\left(\frac{\sqrt{2}\,(1-\sin\theta)}{\sqrt{2}\cos\theta}\right) d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\frac{1-\sin\theta}{\cos\theta}\right) d\theta\right]$$

But (following the hint)

$$\frac{1-\sin\theta}{\cos\theta} = \frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{1-\left[1-2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right]}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} \quad \text{[half-angle formulas]}$$

$$= \frac{2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} = \tan\left(\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right)$$

Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right)\right) d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} \theta\,d\theta + \int_{\pi/6}^{\pi/2} \left[\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right] d\theta\right] = 4\left(\left[\frac{\theta^2}{2}\right]_0^{\pi/6} + \left[\frac{\pi\theta}{4}-\frac{\theta^2}{4}\right]_{\pi/6}^{\pi/2}\right) = 4\left(\frac{3\pi^2}{72}\right) = \frac{\pi^2}{6}$$

7. (a) Since $|xyz| < 1$ except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum\limits_{n=0}^{\infty}(xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = \frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

(b) Since $|-xyz| < 1$, except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1+xyz} = \sum\limits_{n=0}^{\infty}(-xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\sum_{n=0}^{\infty}(-xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] = \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

To evaluate this sum, we first write out a few terms: $s = 1-\dfrac{1}{2^3}+\dfrac{1}{3^3}-\dfrac{1}{4^3}+\dfrac{1}{5^3}-\dfrac{1}{6^3} \approx 0.8998$. Notice that $a_7 = \dfrac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 12.5 [ET 11.5], we have $|s-s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

8. $\displaystyle\int_0^\infty \frac{\arctan \pi x - \arctan x}{x}\,dx = \int_0^\infty \left[\frac{\arctan yx}{x}\right]_{y=1}^{y=\pi} dx = \int_0^\infty \int_1^\pi \frac{1}{1+y^2x^2}\,dy\,dx = \int_1^\pi \int_0^\infty \frac{1}{1+y^2x^2}\,dx\,dy$

$$= \int_1^\pi \lim_{t\to\infty}\left[\frac{\arctan yx}{y}\right]_{x=0}^{x=t} dy = \int_1^\pi \frac{\pi}{2y}\,dy = \frac{\pi}{2}\left[\ln y\right]_1^\pi = \frac{\pi}{2}\ln \pi$$

9. (a) $x = r\cos\theta,\ y = r\sin\theta,\ z = z.$ Then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial r}\right]$$
$$= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta$$

Similarly $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x} r\sin\theta + \dfrac{\partial u}{\partial y} r\cos\theta$ and

$\dfrac{\partial^2 u}{\partial \theta^2} = \dfrac{\partial^2 u}{\partial x^2} r^2\sin^2\theta + \dfrac{\partial^2 u}{\partial y^2} r^2\cos^2\theta - 2\dfrac{\partial^2 u}{\partial y\,\partial x} r^2\sin\theta\,\cos\theta - \dfrac{\partial u}{\partial x} r\cos\theta - \dfrac{\partial u}{\partial y} r\sin\theta$. So

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\,\sin\theta + \frac{\partial u}{\partial x}\frac{\cos\theta}{r} + \frac{\partial u}{\partial y}\frac{\sin\theta}{r}$$
$$+ \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial y\,\partial x}\sin\theta\,\cos\theta$$
$$-\frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r} + \frac{\partial^2 u}{\partial z^2}$$
$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

(b) $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$. Then

$\dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \rho} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \rho} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial \rho} = \dfrac{\partial u}{\partial x}\sin\phi\,\cos\theta + \dfrac{\partial u}{\partial y}\sin\phi\,\sin\theta + \dfrac{\partial u}{\partial z}\cos\phi$, and

$$\frac{\partial^2 u}{\partial \rho^2} = \sin\phi\cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial \rho}\right]$$
$$+ \sin\phi\sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial \rho}\right]$$
$$+ \cos\phi\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial z}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial z}\frac{\partial y}{\partial \rho}\right]$$
$$= 2\frac{\partial^2 u}{\partial y\,\partial x}\sin^2\phi\,\sin\theta\,\cos\theta + 2\frac{\partial^2 u}{\partial z\,\partial x}\sin\phi\,\cos\phi\,\cos\theta + 2\frac{\partial^2 u}{\partial y\,\partial z}\sin\phi\,\cos\phi\,\sin\theta$$
$$+ \frac{\partial^2 u}{\partial x^2}\sin^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\phi\sin^2\theta + \frac{\partial^2 u}{\partial z^2}\cos^2\phi$$

Similarly $\dfrac{\partial u}{\partial \phi} = \dfrac{\partial u}{\partial x}\rho\cos\phi\cos\theta + \dfrac{\partial u}{\partial y}\rho\cos\phi\sin\theta - \dfrac{\partial u}{\partial z}\rho\sin\phi$, and

$$\frac{\partial^2 u}{\partial \phi^2} = 2\frac{\partial^2 u}{\partial y\,\partial x}\rho^2\cos^2\phi\,\sin\theta\,\cos\theta - 2\frac{\partial^2 u}{\partial x\,\partial z}\rho^2\sin\phi\,\cos\phi\,\cos\theta$$
$$- 2\frac{\partial^2 u}{\partial y\,\partial z}\rho^2\sin\phi\,\cos\phi\,\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\cos^2\phi\,\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\rho^2\cos^2\phi\,\sin^2\theta$$
$$+ \frac{\partial^2 u}{\partial z^2}\rho^2\sin^2\phi - \frac{\partial u}{\partial x}\rho\sin\phi\,\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\,\sin\theta - \frac{\partial u}{\partial z}\rho\cos\phi$$

And $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}\,\rho\sin\phi\sin\theta + \dfrac{\partial u}{\partial y}\,\rho\sin\phi\cos\theta$, while

$$\frac{\partial^2 u}{\partial \theta^2} = -2\,\frac{\partial^2 u}{\partial y\,\partial x}\,\rho^2\sin^2\phi\,\cos\theta\,\sin\theta + \frac{\partial^2 u}{\partial x^2}\,\rho^2\sin^2\phi\,\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\,\rho^2\sin^2\phi\cos^2\theta - \frac{\partial u}{\partial x}\,\rho\sin\phi\,\cos\theta - \frac{\partial u}{\partial y}\,\rho\sin\phi\,\sin\theta$$

Therefore

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2} \\
&= \frac{\partial^2 u}{\partial x^2}\left[(\sin^2\phi\,\cos^2\theta) + (\cos^2\phi\,\cos^2\theta) + \sin^2\theta\right] \\
&\quad + \frac{\partial^2 u}{\partial y^2}\left[(\sin^2\phi\,\sin^2\theta) + (\cos^2\phi\,\sin^2\theta) + \cos^2\theta\right] + \frac{\partial^2 u}{\partial z^2}\left[\cos^2\phi + \sin^2\phi\right] \\
&\quad + \frac{\partial u}{\partial x}\left[\frac{2\sin^2\phi\,\cos\theta + \cos^2\phi\,\cos\theta - \sin^2\phi\,\cos\theta - \cos\theta}{\rho\sin\phi}\right] \\
&\quad + \frac{\partial u}{\partial y}\left[\frac{2\sin^2\phi\,\sin\theta + \cos^2\phi\,\sin\theta - \sin^2\phi\,\sin\theta - \sin\theta}{\rho\sin\phi}\right]
\end{aligned}$$

But $2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta = (\sin^2\phi + \cos^2\phi - 1)\,\cos\theta = 0$ and similarly the coefficient of $\partial u/\partial y$ is 0. Also $\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\theta = \cos^2\theta\,(\sin^2\phi + \cos^2\phi) + \sin^2\theta = 1$, and similarly the coefficient of $\partial^2 u/\partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

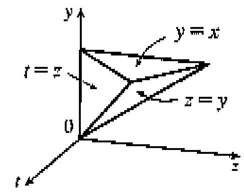
10. (a) Consider a polar division of the disk, similar to that in Figure 16.4.4 [ET 15.4.4], where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$, $0 = r_1 < r_2 < \cdots < r_m = R$, and where the polar subrectangle R_{ij}, as well as r_i^*, θ_j^*, Δr and $\Delta\theta$ are the same as in that figure. Thus $\Delta A_i = r_i^*\,\Delta r\,\Delta\theta$. The mass of R_{ij} is $\rho\,\Delta A_i$, and its distance from m is $s_{ij} \approx \sqrt{(r_i^*)^2 + d^2}$. According to Newton's Law of Gravitation, the force of attraction experienced by m due to this polar subrectangle is in the direction from m towards R_{ij} and has magnitude $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}$. The symmetry of the lamina with respect to the x- and y-axes and the position of m are such that all horizontal components of the gravitational force cancel, so that the total force is simply in the z-direction. Thus, we need only be concerned with the components of this vertical force; that is, $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}\sin\alpha$, where α is the angle between the origin, r_i^* and the mass m. Thus $\sin\alpha = \dfrac{d}{s_{ij}}$ and the previous result becomes $\dfrac{Gm\rho d\,\Delta A_i}{s_{ij}^3}$. The total attractive force is just the Riemann sum

$\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d\,\Delta A_i}{s_{ij}^3} = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Gm\rho d(r_i^*)\,\Delta r\,\Delta\theta}{[(r_i^*)^2 + d^2]^{3/2}}$ which becomes $\displaystyle\int_0^R\int_0^{2\pi}\frac{Gm\rho d}{(r^2+d^2)^{3/2}}\,r\,d\theta\,dr$ as $m \to \infty$ and $n \to \infty$. Therefore,

$$F = 2\pi Gm\rho d\int_0^R \frac{r}{(r^2+d^2)^{3/2}}\,dr = 2\pi Gm\rho d\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^R = 2\pi Gm\rho d\left(\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right)$$

(b) This is just the result of part (a) in the limit as $R \to \infty$. In this case $\dfrac{1}{\sqrt{R^2+d^2}} \to 0$, and we are left with

$$F = 2\pi Gm\rho d\left(\frac{1}{d} - 0\right) = 2\pi Gm\rho.$$



11. $\int_0^x \int_0^y \int_0^z f(t)\, dt\, dz\, dy = \iiint_E f(t)\, dV$, where

$E = \{(t, z, y) \mid 0 \leq t \leq z, 0 \leq z \leq y, 0 \leq y \leq x\}$.

If we let D be the projection of E on the yt-plane then

$D = \{(y, t) \mid 0 \leq t \leq x, t \leq y \leq x\}$. And we see from the diagram

that $E = \{(t, z, y) \mid t \leq z \leq y, t \leq y \leq x, 0 \leq t \leq x\}$. So

$$\begin{aligned}
\int_0^x \int_0^y \int_0^z f(t)\, dt\, dz\, dy &= \int_0^x \int_t^x \int_t^y f(t)\, dz\, dy\, dt = \int_0^x \left[\int_t^x (y - t) f(t)\, dy\right] dt \\
&= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right) f(t)\right]_{y=t}^{y=x} dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right] f(t)\, dt \\
&= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right] f(t)\, dt = \int_0^x \left(\tfrac{1}{2}x^2 - 2tx + t^2\right) f(t)\, dt \\
&= \tfrac{1}{2} \int_0^x (x - t)^2 f(t)\, dt
\end{aligned}$$