CONCEPT CHECK

1. (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

   (b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy-plane. Also, we can use an arrow diagram such as Figure 1 in Section 15.1 [ET 14.1].

2. A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.

3. $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f. We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.

4. (a) See Definition 15.2.4 [ET 14.2.4].

   (b) If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.

5. (a) See (2) and (3) in Section 15.3 [ET 14.3].

   (b) See "Interpretations of Partial Derivatives" on page 917 [ET 881].

   (c) To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

6. See the statement of Clairaut's Theorem on page 921 [ET 885].

7. (a) See (2) in Section 15.4 [ET 14.4]

   (b) See (19) and the preceding discussion in Section 15.6 [ET 14.6].

8. See (3) and (4) and the accompanying discussion in Section 15.4 [ET 14.4]. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b). Thus it is the linear function which best approximates f near (a, b).

9. (a) See Definition 15.4.7 [ET 14.4.7].

   (b) Use Theorem 15.4.8 [ET 14.4.8].

10. See (10) and the associated discussion in Section 15.4 [ET 14.4].

11. See (2) and (3) in Section 15.5 [ET 14.5].

12. See (7) and the preceding discussion in Section 15.5 [ET 14.5].

13. (a) See Definition 15.6.2 [ET 14.6.2]. We can interpret it as the rate of change of f at (x_0, y_0) in the direction of $\mathbf{u}$. Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction $\mathbf{u}$, the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the slope of the tangent line to C at P. (See Figure 5 in Section 15.6 [ET 14.6].)

    (b) See Theorem 15.6.3 [ET 14.6.3].

14. (a) See (8) and (13) in Section 15.6 [ET 14.6].

    (b) $D_{\mathbf{u}}\, f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ or $D_{\mathbf{u}}\, f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

    (c) The gradient vector of a function points in the direction of maximum rate of increase of the function. On a graph of the function, the gradient points in the direction of steepest ascent.

15. (a) f has a local maximum at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b).

    (b) f has an absolute maximum at (a, b) if $f(x, y) \le f(a, b)$ for all points (x, y) in the domain of f.

(c) f has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b).

(d) f has an absolute minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in the domain of f.

(e) f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.

16. (a) By Theorem 15.7.2 [ET 14.7.2], if f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

(b) A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.

17. See (3) in Section 15.7 [ET 14.7]

18. (a) See Figure 11 and the accompanying discussion in Section 15.7 [ET 14.7].

(b) See Theorem 15.7.8 [ET 14.7.8].

(c) See the procedure outlined in (9) in Section 15.7 [ET 14.7].

19. See the discussion beginning on page 970 [ET 934]; see "Two Constraints" on page 974 [ET 938].

TRUE-FALSE QUIZ

1. True. $f_y(a, b) = \lim_{h \to 0} \dfrac{f(a, b + h) - f(a, b)}{h}$ from Equation 15.3.3 [ET 14.3.3]. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a, b) = \lim_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$.

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y \, \partial x}$.

4. True. From Equation 15.6.14 [ET 14.6.14] we get $D_{\mathbf{k}} f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.

5. False. See Example 15.2.3 [ET 14.2.3].

6. False. See Exercise 15.4.46(a) [ET 14.4.46(a)].

7. True. If f has a local minimum and f is differentiable at (a, b) then by Theorem 15.7.2 [ET 14.7.2], $f_x(a, b) = 0$ and $f_y(a, b) = 0$, so $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = \langle 0, 0 \rangle = \mathbf{0}$.

8. False. If f is not continuous at $(2, 5)$, then we can have $\lim_{(x,y) \to (2,5)} f(x, y) \neq f(2, 5)$. (See Example 15.2.7 [ET 14.2.7].)

9. False. $\nabla f(x, y) = \langle 0, 1/y \rangle$.

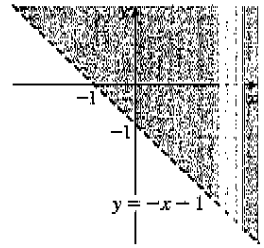
10. True. This is part (c) of the Second Derivatives Test (15.7.3) [ET (14.7.3)].

11. True. $\nabla f = \langle \cos x, \cos y \rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos \theta| \leq 1$, so $|\nabla f| \leq \sqrt{2}$. Now $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u} = |\nabla f| \, |\mathbf{u}| \cos \theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}} f(x, y)| \leq \sqrt{2} \cdot 1 \cdot 1 = \sqrt{2}$.

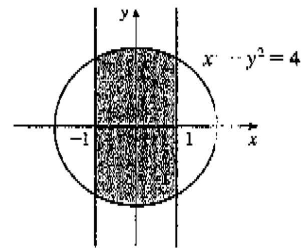
12. False. See Exercise 15.7.37 [ET 14.7.37].

EXERCISES

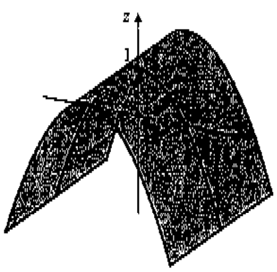
1. $\ln(x + y + 1)$ is defined only when $x + y + 1 > 0 \quad \Rightarrow \quad y > -x - 1$, so the domain of f is $\{(x, y) \mid y > -x - 1\}$, all those points above the line $y = -x - 1$.



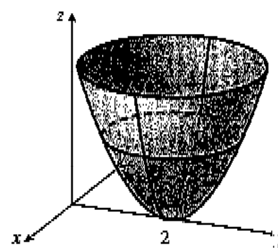
2. $\sqrt{4 - x^2 - y^2}$ is defined only when $4 - x^2 - y^2 \geq 0 \quad \Leftrightarrow \quad x^2 + y^2 \leq 4$, and $\sqrt{1 - x^2}$ is defined only when $1 - x^2 \geq 0 \quad \Leftrightarrow \quad -1 \leq x \leq 1$, so the domain of f is $\left\{(x, y) \mid -1 \leq x \leq 1, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\right\}$, which consists of those points on or inside the circle $x^2 + y^2 = 4$ for $-1 \leq x \leq 1$.



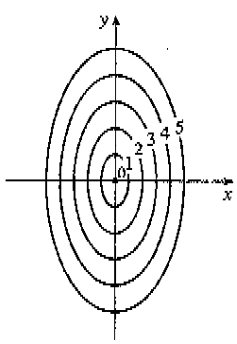
3. $z = f(x, y) = 1 - y^2$, a parabolic cylinder



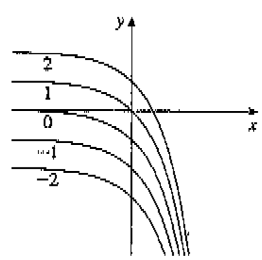
4. $z = f(x, y) = x^2 + (y - 2)^2$, a circular paraboloid with vertex $(0, 2, 0)$ and axis parallel to the z-axis

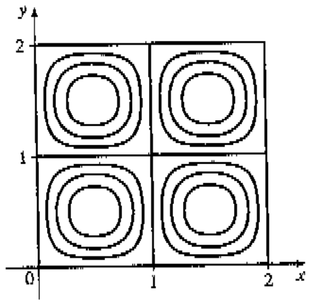


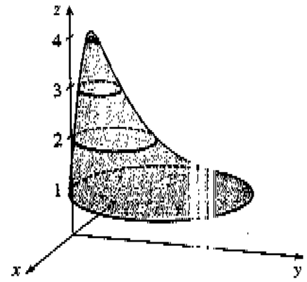
5. The level curves are $\sqrt{4x^2 + y^2} = k$ or $4x^2 + y^2 = k^2$, $k \geq 0$, a family of ellipses.



6. The level curves are $e^x + y = k$ or $y = -e^x + k$, a family of exponential curves.



7. 

8. 

9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1,1)$, we use direct substitution to evaluate the limit: $\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2} = \frac{2(1)(1)}{1^2+2(1)^2} = \frac{2}{3}$.

10. As $(x,y) \to (0,0)$ along the x-axis, $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ along this line. But $f(x,x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x,y) \to (0,0)$ along the line $x = y$, $f(x,y) \to \frac{2}{3}$. Thus the limit doesn't exist.

11. (a) $T_x(6,4) = \lim_{h\to 0} \frac{T(6+h,4)-T(6,4)}{h}$, so we can approximate $T_x(6,4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6,4) \approx \frac{T(8,4)-T(6,4)}{2} = \frac{86-80}{2} = 3$,

$T_x(6,4) \approx \frac{T(4,4)-T(6,4)}{-2} = \frac{72-80}{-2} = 4$. Averaging these values, we estimate $T_x(6,4)$ to be approximately $3.5°\text{C/m}$. Similarly, $T_y(6,4) =$ [...], which we can approximate with $h = \pm 2$:

$T_y(6,4) \approx \frac{T(6,6)-T(6,4)}{2} =$ [...] $\approx \frac{T(6,2)-T(6,4)}{-2} = \frac{87-80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6,4)$ to [...].

Mary -
It was good to see you last month - thank you for making the effort to come see us:-)
A friend talked me into going to a conference in DC at the end of November. Would it be

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Eq[...] $D_\mathbf{u} T(6,4) = \nabla T(6,4) \cdot \mathbf{u} = T_x(6,4)\frac{1}{\sqrt{2}} + T_y(6,4)\frac{1}{\sqrt{2}}$.

Using our estimates from part (a), we have $D_\mathbf{u} T(6,4) \approx (3.5)\frac{1}{\sqrt{2}} + (-3.0)\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6,4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35°\text{C/m}$.

Alternatively, we can use Definition 15.6.2 [ET 14.6.2]: $D_\mathbf{u} T(6,4) = \lim_{h\to 0} \frac{T\left(6+h\frac{1}{\sqrt{2}}, 4+h\frac{1}{\sqrt{2}}\right) - T(6,4)}{h}$,

which we can estimate with $h = \pm 2\sqrt{2}$. Then $D_\mathbf{u} T(6,4) \approx \frac{T(8,6)-T(6,4)}{2\sqrt{2}} = \frac{80-80}{2\sqrt{2}} = 0$,

$D_\mathbf{u} T(6,4) \approx \frac{T(4,2)-T(6,4)}{-2\sqrt{2}} = \frac{74-80}{-2\sqrt{2}} = \frac{3}{\sqrt{2}}$. Averaging these values, we have $D_\mathbf{u} T(6,4) \approx \frac{3}{2\sqrt{2}} \approx 1.1°\text{C/m}$.

(c) $T_{xy}(x,y) = \frac{\partial}{\partial y}[T_x(x,y)] = \lim_{h\to 0} \frac{T_x(x,y+h)-T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim_{h\to 0} \frac{T_x(6,4+h)-T_x(6,4)}{h}$ which we can estimate with $h = \pm 2$. We have $T_x(6,4) \approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h = \pm 2$ and the values given in the table, we have

$T_x(6,6) \approx \frac{T(8,6)-T(6,6)}{2} = \frac{80-75}{2} = 2.5$, $T_x(6,6) \approx \frac{T(4,6)-T(6,6)}{-2} = \frac{68-75}{-2} = 3.5$.

Averaging these values, we estimate $T_x(6,6) \approx 3.0$. Similarly,

$$T_x(6,2) \approx \frac{T(8,2) - T_x(6,2)}{2} = \frac{90 - 87}{2} = 1.5,\ T_x(6,2) \approx \frac{T(4,2) - T(6,2)}{-2} = \frac{74 - 87}{-2} = 6.5.$$

Averaging these values, we estimate $T_x(6,2) \approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$$T_{xy}(6,4) \approx \frac{T_x(6,6) - T_x(6,4)}{2} = \frac{3.0 - 3.5}{2} = -0.25,\ T_{xy}(6,4) \approx \frac{T_x(6,2) - T_x(6,4)}{-2} = \frac{4.0 - 3.5}{-2} = -0.25.$$

Averaging these values, we have $T_{xy}(6,4) \approx -0.25$.

12. From the table, $T(6,4) = 80$, and from Exercise 11 we estimated $T_x(6,4) \approx 3.5$ and $T_y(6,4) \approx -3.0$. The linear approximation then is

$$T(x,y) \approx T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4) \approx 80 + 3.5(x-6) - 3(y-4) = 3.5x - 3y + 71$$

Thus at the point $(5, 3.8)$, we can use the linear approximation to estimate $T(5, 3.8) \approx 3.5(5) - 3(3.8) + 71 \approx 77.1°C$.

13. $f(x,y) = \sqrt{2x+y^2} \quad\Rightarrow\quad f_x = \frac{1}{2}(2x+y^2)^{-1/2}(2) = \dfrac{1}{\sqrt{2x+y^2}},\ f_y = \frac{1}{2}(2x+y^2)^{-1/2}(2y) = \dfrac{y}{\sqrt{2x+y^2}}$

14. $u = e^{-r}\sin 2\theta \quad\Rightarrow\quad u_r = -e^{-r}\sin 2\theta,\ u_\theta = 2e^{-r}\cos 2\theta$

15. $g(u,v) = u\tan^{-1} v \quad\Rightarrow\quad g_u = \tan^{-1} v,\ g_v = \dfrac{u}{1+v^2}$

16. $w = \dfrac{x}{y-z} \quad\Rightarrow\quad w_x = \dfrac{1}{y-z},\ w_y = x(-1)(y-z)^{-2} = -\dfrac{x}{(y-z)^2},\ w_z = x(-1)(y-z)^{-2}(-1) = \dfrac{x}{(y-z)^2}$

17. $T(p,q,r) = p\ln(q+e^r) \quad\Rightarrow\quad T_p = \ln(q+e^r),\ T_q = \dfrac{p}{q+e^r},\ T_r = \dfrac{pe^r}{q+e^r}$

18. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S-35) + 0.016D \quad\Rightarrow$
$\partial C/\partial T = 4.6 - 0.11T + 0.00087T^2 - 0.01(S-35)$, $\partial C/\partial S = 1.34 - 0.01T$, and $\partial C/\partial D = 0.016$. When $T = 10$, $S = 35$, and $D = 100$ we have $\partial C/\partial T = 4.6 - 0.11(10) + 0.00087(10)^2 - 0.01(35-35) \approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S = 1.34 - 0.01(10) = 1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D = 0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

19. $f(x,y) = 4x^3 - xy^2 \quad\Rightarrow\quad f_x = 12x^2 - y^2,\ f_y = -2xy,\ f_{xx} = 24x,\ f_{yy} = -2x,\ f_{xy} = f_{yx} = -2y$

20. $z = xe^{-2y} \quad\Rightarrow\quad z_x = e^{-2y},\ z_y = -2xe^{-2y},\ z_{xx} = 0,\ z_{yy} = 4xe^{-2y},\ z_{xy} = z_{yx} = -2e^{-2y}$

21. $f(x,y,z) = x^k y^l z^m \quad\Rightarrow\quad f_x = kx^{k-1}y^l z^m,\ f_y = lx^k y^{l-1} z^m,\ f_z = mx^k y^l z^{m-1},\ f_{xx} = k(k-1)x^{k-2}y^l z^m,$
$f_{yy} = l(l-1)x^k y^{l-2} z^m,\ f_{zz} = m(m-1)x^k y^l z^{m-2},\ f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m,\ f_{xz} = f_{zx} = kmx^{k-1}y^l z^{m-1},$
$f_{yz} = f_{zy} = lmx^k y^{l-1} z^{m-1}$

22. $v = r\cos(s+2t) \quad\Rightarrow\quad v_r = \cos(s+2t),\ v_s = -r\sin(s+2t),\ v_t = -2r\sin(s+2t),\ v_{rr} = 0,\ v_{ss} = -r\cos(s+2t),$
$v_{tt} = -4r\cos(s+2t),\ v_{rs} = v_{sr} = -\sin(s+2t),\ v_{rt} = v_{tr} = -2\sin(s+2t),\ v_{st} = v_{ts} = -2r\cos(s+2t)$

23. $z = xy + xe^{y/x} \Rightarrow \frac{\partial z}{\partial x} = y - \frac{y}{x}e^{y/x} + e^{y/x}$, $\frac{\partial z}{\partial y} = x + e^{y/x}$ and

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(y - \frac{y}{x}e^{y/x} + e^{y/x}\right) + y\left(x + e^{y/x}\right) = xy - ye^{y/x} + xe^{y/x} + xy + ye^{y/x} = xy + xy + xe^{y/x} = xy + z.$

24. $z = \sin(x + \sin t) \Rightarrow \frac{\partial z}{\partial x} = \cos(x + \sin t)$, $\frac{\partial z}{\partial t} = \cos(x + \sin t)\cos t$,

$\frac{\partial^2 z}{\partial x \partial t} = -\sin(x + \sin t)\cos t$, $\frac{\partial^2 z}{\partial x^2} = -\sin(x + \sin t)$ and

$\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial t} = \cos(x + \sin t)\left[-\sin(x + \sin t)\cos t\right] = \cos(x + \sin t)(\cos t)\left[-\sin(x + \sin t)\right] = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}.$

25. (a) $z_x = 6x + 2 \Rightarrow z_x(1, -2) = 8$ and $z_y = -2y \Rightarrow z_y(1, -2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x - 1) + 4(y + 2)$ or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1, -2, 1)$ is $\langle 8, 4, -1\rangle$. Then parametric equations for the normal line there are $x = 1 + 8t$, $y = -2 + 4t$, $z = 1 - t$, and symmetric equations are $\frac{x-1}{8} = \frac{y+2}{4} = \frac{z-1}{-1}$.

26. (a) $z_x = e^x \cos y \Rightarrow z_x(0, 0) = 1$ and $z_y = -e^x \sin y \Rightarrow z_y(0, 0) = 0$, so an equation of the tangent plane is $z - 1 = 1(x - 0) + 0(y - 0)$ or $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0, 0, 1)$ is $\langle 1, 0, -1\rangle$. Then parametric equations for the normal line there are $x = t$, $y = 0$, $z = 1 - t$, and symmetric equations are $x = 1 - z$, $y = 0$.

27. (a) Let $F(x, y, z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2, -1, 1) = 4$, $F_y(2, -1, 1) = -4$, $F_z(2, -1, 1) = -6$. From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $4(x - 2) - 4(y + 1) - 6(z - 1) = 0$ or, equivalently, $2x - 2y - 3z = 3$.

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\frac{x-2}{4} = \frac{y+1}{-4} = \frac{z-1}{-6}$.

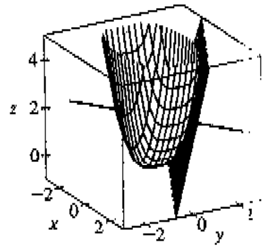
28. (a) Let $F(x, y, z) = xy + yz + zx$. Then $F_x = y + z$, $F_y = x + z$, $F_z = x + y$, so $F_x(1, 1, 1) = F_y(1, 1, 1) = F_z(1, 1, 1) = 2$. From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $2(x - 1) + 2(y - 1) + 2(z - 1) = 0$ or, equivalently, $x + y + z = 3$.

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$ or, equivalently, $x = y = z$.

29. (a) $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + u^2\,\mathbf{j} + v^2\,\mathbf{k}$ and the point $(3, 4, 1)$ corresponds to $u = 2$, $v = 1$. Then $\mathbf{r}_u = \mathbf{i} + 2u\,\mathbf{j} \Rightarrow \mathbf{r}_u(2, 1) = \mathbf{i} + 4\,\mathbf{j}$ and $\mathbf{r}_v = \mathbf{i} + 2v\,\mathbf{k} \Rightarrow \mathbf{r}_v(2, 1) = \mathbf{i} + 2\,\mathbf{j}$. A normal vector to the surface at $(3, 4, 1)$ is $\mathbf{r}_u \times \mathbf{r}_v = 8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}$, so an equation of the tangent plane there is $8(x - 3) - 2(y - 4) - 4(z - 1) = 0$ or equivalently $4x - y - 2z = 6$.

(b) A direction vector for the normal line through $(3, 4, 1)$ is $8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k}$, so a vector equation is $\mathbf{r}(t) = (3\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}) + t\,(8\,\mathbf{i} - 2\,\mathbf{j} - 4\,\mathbf{k})$, and the corresponding parametric equations are $x = 3 + 8t$, $y = 4 - 2t$, $z = 1 - 4t$.

30. Let $f(x, y) = x^2 + y^4$. Then $f_x(x, y) = 2x$ and $f_y(x, y) = 4y^3$, so $f_x(1, 1) = 2$, $f_y(1, 1) = 4$ and an equation of the tangent plane is $z - 2 = 2(x - 1) + 4(y - 1)$ or $2x + 4y - z = 4$. A normal vector to the tangent plane is $\langle 2, 4, -1 \rangle$ so the normal line is given by $\frac{x-1}{2} = \frac{y-1}{4} = \frac{z-2}{-1}$ or $x = 1 + 2t$, $y = 1 + 4t$, $z = 2 - t$.



31. The hyperboloid is a level surface of the function $F(x, y, z) = x^2 + 4y^2 - z^2$, so a normal vector to the surface at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = \langle 2x_0, 8y_0, -2z_0 \rangle$. A normal vector for the plane $2x + 2y + z = 5$ is $\langle 2, 2, 1 \rangle$. For the planes to be parallel, we need the normal vectors to be parallel, so $\langle 2x_0, 8y_0, -2z_0 \rangle = k\langle 2, 2, 1 \rangle$, or $x_0 = k$, $y_0 = \frac{1}{4}k$, and $z_0 = -\frac{1}{2}k$. But $x_0^2 + 4y_0^2 - z_0^2 = 4 \Rightarrow k^2 + \frac{1}{4}k^2 - \frac{1}{4}k^2 = 4 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$. So there are two such points: $\left(2, \frac{1}{2}, -1\right)$ and $\left(-2, -\frac{1}{2}, 1\right)$.

32. $u = \ln(1 + se^{2t}) \Rightarrow du = \frac{\partial u}{\partial s}\,ds + \frac{\partial u}{\partial t}\,dt = \frac{e^{2t}}{1 + se^{2t}}\,ds + \frac{2se^{2t}}{1 + se^{2t}}\,dt$

33. $f(x, y, z) = x^3\sqrt{y^2 + z^2} \Rightarrow f_x(x, y, z) = 3x^2\sqrt{y^2 + z^2}$, $f_y(x, y, z) = \frac{yx^3}{\sqrt{y^2 + z^2}}$, $f_z(x, y, z) = \frac{zx^3}{\sqrt{y^2 + z^2}}$, so $f(2, 3, 4) = 8(5) = 40$, $f_x(2, 3, 4) = 3(4)\sqrt{25} = 60$, $f_y(2, 3, 4) = \frac{3(8)}{\sqrt{25}} = \frac{24}{5}$, and $f_z(2, 3, 4) = \frac{4(8)}{\sqrt{25}} = \frac{32}{5}$. Then the linear approximation of f at $(2, 3, 4)$ is

$$\begin{aligned} f(x, y, z) &\approx f(2, 3, 4) + f_x(2, 3, 4)(x - 2) + f_y(2, 3, 4)(y - 3) + f_z(2, 3, 4)(z - 4) \\ &= 40 + 60(x - 2) + \tfrac{24}{5}(y - 3) + \tfrac{32}{5}(z - 4) = 60x + \tfrac{24}{5}y + \tfrac{32}{5}z - 120 \end{aligned}$$

Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2} = f(1.98, 3.01, 3.97) \approx 60(1.98) + \frac{24}{5}(3.01) + \frac{32}{5}(3.97) - 120 = 38.656$.

34. (a) $dA = \frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated area is about $dA = 6(0.002) + \frac{5}{2}(0.002) = 0.017\ \text{m}^2$ or $170\ \text{cm}^2$.

(b) $z = \sqrt{x^2 + y^2}$, $dz = \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy$ and $|\Delta x| \le 0.002$, $|\Delta y| \le 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz = \frac{5}{13}(0.002) + \frac{12}{13}(0.002) = \frac{0.17}{65} \approx 0.0026$ m or 0.26 cm.

35. $\frac{du}{dp} = \frac{\partial u}{\partial x}\frac{dx}{dp} + \frac{\partial u}{\partial y}\frac{dy}{dp} + \frac{\partial u}{\partial z}\frac{dz}{dp} = 2xy^3(1 + 6p) + 3x^2y^2(pe^p + e^p) + 4z^3(p\cos p + \sin p)$

36. $\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} = \left(2x\sin y + y^2e^{xy}\right)(1) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(t)$.

$s = 0, t = 1 \Rightarrow x = 2, y = 0$, so $\frac{\partial v}{\partial s} = 0 + (4 + 1)(1) = 5$.

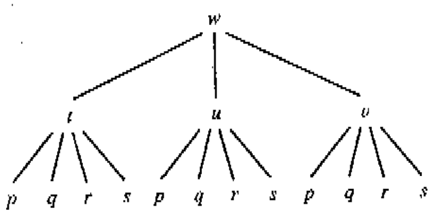
$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} = \left(2x\sin y + y^2e^{xy}\right)(2) + \left(x^2\cos y + xye^{xy} + e^{xy}\right)(s) = 0 + 0 = 0$.

37. By the Chain Rule, $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$. When $s = 1$ and $t = 2$, $x = g(1,2) = 3$ and $y = h(1,2) = 6$, so

$\frac{\partial z}{\partial s} = f_x(3,6)g_s(1,2) + f_y(3,6)\,h_s(1,2) = (7)(-1) + (8)(-5) = -47$. Similarly, $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$, so

$\frac{\partial z}{\partial t} = f_x(3,6)g_t(1,2) + f_y(3,6)\,h_t(1,2) = (7)(4) + (8)(10) = 108$.

38.



Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p} \qquad \frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

39. $\frac{\partial z}{\partial x} = 2xf'(x^2 - y^2)$, $\quad \frac{\partial z}{\partial y} = 1 - 2yf'(x^2 - y^2)$ $\quad \left[\text{where } f' = \frac{df}{d(x^2 - y^2)}\right]$. Then

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 2xyf'(x^2 - y^2) + x - 2xyf'(x^2 - y^2) = x$.

40. $A = \frac{1}{2}xy\sin\theta$, $dx/dt = 3$, $dy/dt = -2$, $d\theta/dt = 0.05$, and $\frac{dA}{dt} = \frac{1}{2}\left[(y\sin\theta)\frac{dx}{dt} + (x\sin\theta)\frac{dy}{dt} + (xy\cos\theta)\frac{d\theta}{dt}\right]$.

So when $x = 40$, $y = 50$ and $\theta = \frac{\pi}{6}$, $\frac{dA}{dt} = \frac{1}{2}\left[(25)(3) + (20)(-2) + \left(1000\sqrt{3}\right)(0.05)\right] = \frac{35 + 50\sqrt{3}}{2} \approx 60.8 \text{ in}^2\text{/s}$.

41. $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}y + \frac{\partial z}{\partial v}\frac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{2y}{x^3}\frac{\partial z}{\partial v} + \frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial v} + y\left(\frac{\partial^2 z}{\partial u^2}y + \frac{\partial^2 z}{\partial v\,\partial u}\frac{-y}{x^2}\right) + \frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\frac{-y}{x^2} + \frac{\partial^2 z}{\partial u\,\partial v}y\right)\\ &= \frac{2y}{x^3}\frac{\partial z}{\partial v} + y^2\frac{\partial^2 z}{\partial u^2} - \frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial z}{\partial v}$ and

$$\frac{\partial^2 z}{\partial y^2} = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = x\left(\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial v\,\partial u}\frac{1}{x}\right) + \frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{x} + \frac{\partial^2 z}{\partial u\,\partial v}x\right) = x^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}$$

Thus

$$\begin{aligned}x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} &= \frac{2y}{x}\frac{\partial z}{\partial v} + x^2y^2\frac{\partial^2 z}{\partial u^2} - 2y^2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2y^2\frac{\partial^2 z}{\partial u^2} - 2y^2\frac{\partial^2 z}{\partial u\,\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}\\ &= \frac{2y}{x}\frac{\partial z}{\partial v} - 4y^2\frac{\partial^2 z}{\partial u\,\partial v} = 2v\frac{\partial z}{\partial v} - 4uv\frac{\partial^2 z}{\partial u\,\partial v}\end{aligned}$$

since $y = xv = \frac{uv}{y}$ or $y^2 = uv$.

42. $F(x,y,z) = e^{xyz} - yz^4 - x^2z^3 = 0$, so $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yze^{xyz} - 2xz^3}{xye^{xyz} - 4yz^3 - 3x^2z^2} = \frac{2xz^3 - yze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2z^2}$ and

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xze^{xyz} - z^4}{xye^{xyz} - 4yz^3 - 3x^2z^2} = \frac{z^4 - xze^{xyz}}{xye^{xyz} - 4yz^3 - 3x^2z^2}$.

43. $\nabla f = \left\langle z^2\sqrt{y}\,e^{x\sqrt{y}}, \dfrac{xz^2e^{x\sqrt{y}}}{2\sqrt{y}}, 2ze^{x\sqrt{y}}\right\rangle = ze^{x\sqrt{y}}\left\langle z\sqrt{y}, \dfrac{xz}{2\sqrt{y}}, 2\right\rangle$

44. (a) By Theorem 15.6.15 [ET 14.6.15], the maximum value of the directional derivative occurs when **u** has the same direction as the gradient vector.

(b) It is a minimum when **u** is in the direction opposite to that of the gradient vector (that is, **u** is in the direction of $-\nabla f$), since $D_{\mathbf{u}}\,f = |\nabla f|\cos\theta$ (see the proof of Theorem 15.6.15 [ET 14.6.15]) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when **u** is perpendicular to the gradient vector, since then $D_{\mathbf{u}}\,f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}\,f = |\nabla f|\cos\theta = \frac{1}{2}|\nabla f| \quad\Leftrightarrow\quad \cos\theta = \frac{1}{2} \quad\Leftrightarrow\quad \theta = \frac{\pi}{3}$.

45. $\nabla f = \langle 1/\sqrt{x}, -2y\rangle$, $\nabla f(1,5) = \langle 1, -10\rangle$, $\mathbf{u} = \frac{1}{5}\langle 3, -4\rangle$. Then $D_{\mathbf{u}}\,f(1,5) = \frac{43}{5}$.

46. $\nabla f = \left\langle 2xy + \sqrt{1+z}, x^2, x/\left(2\sqrt{1+z}\right)\right\rangle$, $\nabla f(1,2,3) = \left\langle 6, 1, \frac{1}{4}\right\rangle$, $\mathbf{u} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle$. Then $D_{\mathbf{u}}\,f(1,2,3) = \frac{25}{6}$.

47. $\nabla f = \left\langle 2xy, x^2 + 1/\left(2\sqrt{y}\right)\right\rangle$, $|\nabla f(2,1)| = \left|\left\langle 4, \frac{9}{2}\right\rangle\right|$. Thus the maximum rate of change of f at $(2,1)$ is $\frac{\sqrt{145}}{2}$ in the direction $\left\langle 4, \frac{9}{2}\right\rangle$.

48. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy}\rangle$, $\nabla f(0,1,2) = \langle 2, 0, 1\rangle$ is the direction of most rapid increase while the rate is $|\langle 2, 0, 1\rangle| = \sqrt{5}$.

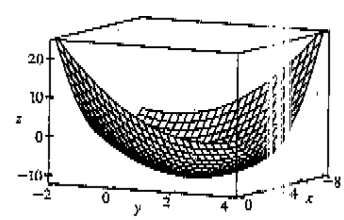
49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8} = \frac{5}{8} = 0.625$ knot/mi.

50. The surfaces are $f(x,y,z) = z - 2x^2 + y^2 = 0$ and $g(x,y,z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2,2,4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line. $\nabla f(x,y,z) = \langle -4x, 2y, 1\rangle \Rightarrow \nabla f(-2,2,4) = \langle 8,4,1\rangle$, $\nabla g(x,y,z) = \langle 0,0,1\rangle \quad\Rightarrow\quad \nabla g(-2,2,4) = \langle 0,0,1\rangle$. Hence

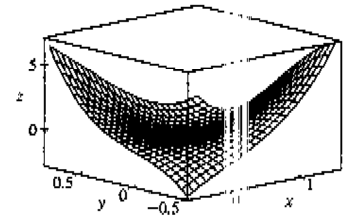
$$\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\,\mathbf{i} - 8\,\mathbf{j}.$$

Thus, parametric equations are: $x = -2 + 4t$, $y = 2 - 8t$, $z = 4$.

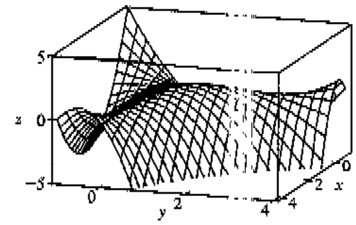
51. $f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10 \quad\Rightarrow\quad f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4, 1)$ and $f_{xx}(-4,1) > 0$, $D(-4,1) = 3 > 0$, so $f(-4,1) = -11$ is a local minimum.



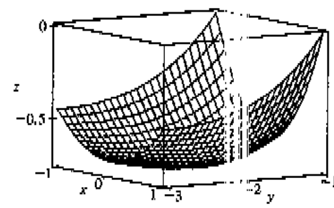
52. $f(x,y) = x^3 - 6xy + 8y^3 \Rightarrow f_x = 3x^2 - 6y,\ f_y = -6x + 24y^2$, $f_{xx} = 6x,\ f_{yy} = 48y,\ f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x(x^3 - 1) = 0$, so the critical points are $(0,0)$, $(1, \frac{1}{2})$. $D(0,0) = -36 < 0$ so $(0,0)$ is a saddle point while $f_{xx}(1, \frac{1}{2}) = 6 > 0$ and $D(1, \frac{1}{2}) = 108 > 0$ so $f(1, \frac{1}{2}) = -1$ is a local minimum.



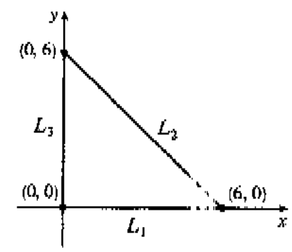
53. $f(x,y) = 3xy - x^2y - xy^2 \Rightarrow f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy,\ f_{xx} = -2y,\ f_{yy} = -2x,\ f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0,0)$, $(3,0)$, $(0,3)$ and $(1,1)$. $D(0,0) = D(3,0) = D(0,3) = -9 < 0$ so $(0,0)$, $(3,0)$, and $(0,3)$ are saddle points. $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = -2 < 0$, so $f(1,1) = 1$ is a local maximum.



54. $f(x,y) = (x^2 + y)e^{y/2} \Rightarrow f_x = 2xe^{y/2},\ f_y = e^{y/2}(2 + x^2 + y)/2$, $f_{xx} = 2e^{y/2},\ f_{yy} = e^{y/2}(4 + x^2 + y)/4,\ f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0,-2) > 0$, $D(0,-2) = e^{-2} - 0 > 0$ so $f(0,-2) = -2/e$ is a local minimum.



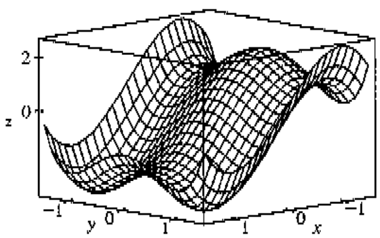
55. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3,\ f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2,0)$ isn't inside D. Thus the only critical point inside D is $(1,2)$ and $f(1,2) = 4$. Secondly we consider the boundary of D. On L_1: $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2: $x = -y + 6$ and

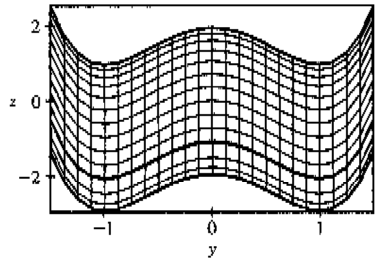


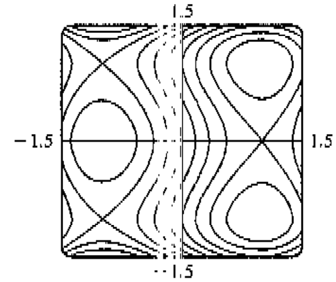
$f(-y + 6, y) = y^2(6 - y)(-2) = -2(6y^2 - y^3)$ which has critical pointsat $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3: $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.

56. Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0,0)$, $(0,\pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0,0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0,\pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x,y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0,\pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f(0,\pm 1) = 2e^{-1}$ and the absolute minimum is $f(0,0) = 0$.

57. $f(x,y) = x^3 - 3x + y^4 - 2y^2$



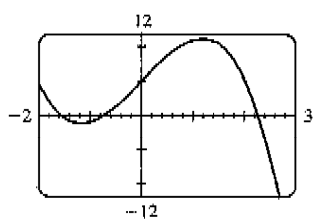


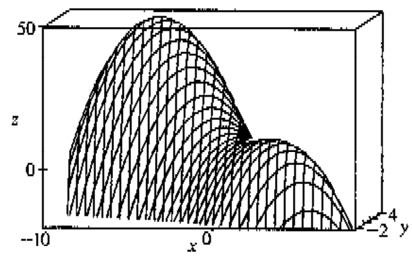


From the graphs, it appears that f has a local maximum $f(-1,0) \approx 2$, local minima $f(1,\pm 1) \approx -3$, and saddle points at $(-1,\pm 1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \Leftrightarrow$ $y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1,0) = 24 > 0$ and $f_{xx}(-1,0) = -6 < 0$ indicating a local maximum $f(-1,0) = 2$; $D(1,\pm 1) = 48 > 0$ and $f_{xx}(1,\pm 1) = 6 > 0$ indicating local minima $f(1,\pm 1) = -3$; and $D(-1,\pm 1) = -48$ and $D(1,0) = -24$, indicating saddle points.

58. $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4 \Rightarrow f_x(x,y) = -4x - 8y$, $f_y(x,y) = 10 - 8x - 4y^3$. Now $f_x(x,y) = 0 \Rightarrow$ $x = -2x$, and substituting this into $f_y(x,y) = 0$ gives $10 + 16y - 4y^3 = 0 \Leftrightarrow 5 + 8y - 2y^3 = 0$.





From the first graph, we see that this is true when $y \approx -1.542, -0.717$, or 2.260. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x,y)$ has local maxima $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$

59. $f(x,y) = x^2 y$, $g(x,y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ imply $\lambda = x^2/(2y)$ and $\lambda = y$ if $x \neq 0$ and $y \neq 0$. Hence $x^2 = 2y^2$. Then $x^2 + y^2 = 1$ implies $3y^2 = 1$ so $y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm\sqrt{\frac{2}{3}}$. [Note if $x = 0$ then $x^2 = 2\lambda y$ implies $y = 0$ and $f(0,0) = 0$.] Thus the possible points are $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{3}}\right)$ and the absolute maxima are $f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ while the absolute minima are $f\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$

60. $f(x,y) = 1/x + 1/y$, $g(x,y) = 1/x^2 + 1/y^2 = 1 \Rightarrow \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^{3}$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

61. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + y^2 + z^2 = 3$. $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x, y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} = \dfrac{xy}{2z} \Rightarrow 2y^2z = 2x^2z \Rightarrow$ $y^2 = x^2$, and similarly $2yz^2 = 2x^2y \Rightarrow z^2 = x^2$. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = 3 \Rightarrow$ $x^2 = 1 = y^2 = z^2$. Thus the possible points are $(1,1,\pm1)$, $(1,-1,\pm1)$, $(-1,1,\pm1)$, $(-1,-1,\pm1)$. The absolute maximum is $f(1,1,1) = f(1,-1,-1) = f(-1,1,-1) = f(-1,-1,1) = 1$ and the absolute minimum is $f(1,1,-1) = f(1,-1,1) = f(-1,1,1) = f(-1,-1,-1) = -1$.

62. $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = x - y + 2z = 2 \Rightarrow$ $\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and $2x = \lambda + \mu$ **(1)**, $4y = \lambda - \mu$ **(2)**, $6z = \lambda + 2\mu$ **(3)**, $x + y + z = 1$ **(4)**, $x - y + 2z = 2$ **(5)**. Then six times **(1)** plus three times **(2)** plus two times **(3)** implies $12(x + y + z) = 11\lambda + 7\mu$, so **(4)** gives $11\lambda + 7\mu = 12$. Also six times **(1)** minus three times **(2)** plus four times **(3)** implies $12(x - y + 2z) = 7\lambda + 17\mu$, so **(5)** gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into **(1)**, **(2)**, and **(3)** implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$. Now since $(0,0,1)$ satisfies both constraints and $f(0,0,1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

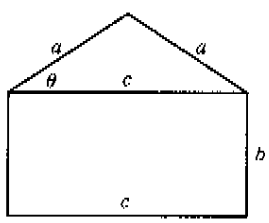
63. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = xy^2z^3 = 2 \Rightarrow \nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle \lambda y^2z^3, 2\lambda xyz^3, 3\lambda xy^2z^2 \rangle$. Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so $2x = \lambda y^2z^3$ **(1)**, $1 = \lambda xz^3$ **(2)**, $2 = 3\lambda xy^2z$ **(3)**. Then **(2)** and **(3)** imply $\dfrac{1}{xz^3} = \dfrac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so $y = \pm z\sqrt{\frac{2}{3}}$. Similarly **(1)** and **(3)** imply $\dfrac{2x}{y^2z^3} = \dfrac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm\frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x,y,z) = 2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3 = 2$ or $z = \pm 3^{1/4}$ and the possible points are $(\pm3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm3^{1/4})$, $(\pm3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm3^{1/4})$. However at each of these points f takes on the same value, $2\sqrt{3}$. But $(2,1,1)$ also satisfies $g(x,y,z) = 2$ and $f(2,1,1) = 6 > 2\sqrt{3}$. Thus f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3 = 2$.

Alternate solution: $g(x,y,z) = xy^2z^3 = 2$ implies $y^2 = \dfrac{2}{xz^3}$, so minimize $f(x,z) = x^2 + \dfrac{2}{xz^3} + z^2$. Then $f_x = 2x - \dfrac{2}{x^2z^3}$, $f_z = -\dfrac{6}{xz^4} + 2z$, $f_{xx} = 2 + \dfrac{4}{x^3z^3}$, $f_{zz} = \dfrac{24}{xz^5} + 2$ and $f_{xz} = \dfrac{6}{x^2z^4}$. Now $f_x = 0$ implies $2x^3z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_y = 0$ implies $-6x^3 + 2x^{-1} = 0$ or $x = \frac{1}{\sqrt[4]{3}}$, so the two critical points are

$\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = (2+4)(2+\frac{24}{3}) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \dfrac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$.

64. $V = xyz$, say x is the length and $x + 2y + 2z \le 108$, $x > 0$, $y > 0$, $z > 0$. First maximize V subject to $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy\rangle = \langle \lambda, 2\lambda, 2\lambda\rangle$ implies $2yz = xz$ or $x = 2y$ and $xz = xy$ or $z = y$. Thus $g(x,y,z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11{,}664$ cubic units. Since $(104, 1, 1)$ also satisfies $g(x,y,z) = 108$ and $V(104,1,1) = 104$ cubic units, $(36, 18, 18)$ gives an absolute maximum of V subject to $g(x,y,z) = 108$. But if $x + 2y + 2z < 108$, there exists $\alpha > 0$ such that $x + 2y + 2z = 108 - \alpha$ and as above $6y = 108 - \alpha$ implies $y = (108-\alpha)/6 = z$, $x = (108-\alpha)/3$ with $V = (108-\alpha)^3/(6^2 \cdot 3) < (108)^3/(6^2\cdot 3) = 11{,}664$. Hence we have shown that the maximum of V subject to $g(x,y,z) \le 108$ is the maximum of V subject to $g(x,y,z) = 108$ (an intuitively obvious fact).

65.



The area of the triangle is $\frac{1}{2}ca\sin\theta$ and the area of the rectangle is bc. Thus, the area of the whole object is $f(a,b,c) = \frac{1}{2}ca\sin\theta + bc$. The perimeter of the object is $g(a,b,c) = 2a + 2b + c = P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2\sin^2\theta + \left(\frac{1}{2}c\right)^2 = a^2 \;\Rightarrow\; \sin\theta = \dfrac{1}{2a}\sqrt{4a^2 - c^2}$. Thus $f(a,b,c) = \dfrac{c}{4}\sqrt{4a^2 - c^2} + bc$. (Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f = \lambda\nabla g$ which gives the following equations: $ca(4a^2 - c^2)^{-1/2} = 2\lambda$ **(1)**, $c = 2\lambda$ **(2)**, $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$ **(3)**, and $2a + 2b + c = P$ **(4)**. From **(2)**, $\lambda = \frac{1}{2}c$ and so **(1)** produces $ca(4a^2 - c^2)^{-1/2} = c \;\Rightarrow\; (4a^2 - c^2)^{1/2} = a \;\Rightarrow\; 4a^2 - c^2 = a^2 \;\Rightarrow\; c = \sqrt{3}\,a$ **(5)**. Similarly, since $\left(4a^2 - c^2\right)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, **(3)** gives $\dfrac{a}{4} - \dfrac{c^2}{4a} + b = \dfrac{c}{2}$, so from **(5)**, $\dfrac{a}{4} - \dfrac{3a}{4} + b = \dfrac{\sqrt{3}\,a}{2} \;\Rightarrow\; -\dfrac{a}{2} - \dfrac{\sqrt{3}\,a}{2} = -b \;\Rightarrow\; b = \dfrac{a}{2}\left(1 + \sqrt{3}\right)$ **(6)**. Substituting **(5)** and **(6)** into **(4)** we get:

$$2a + a\left(1 + \sqrt{3}\right) + \sqrt{3}\,a = P \;\Rightarrow\; 3a + 2\sqrt{3}\,a = P \;\Rightarrow\; a = \frac{P}{3 + 2\sqrt{3}} = \frac{2\sqrt{3} - 3}{3}P$$ and thus

$$b = \frac{\left(2\sqrt{3} - 3\right)\left(1 + \sqrt{3}\right)}{6}P = \frac{3 - \sqrt{3}}{6}P \text{ and } c = \left(2 - \sqrt{3}\right)P.$$

66. (a) $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + f(x(t), y(t))\,\mathbf{k} \;\Rightarrow\; \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j} + \left(f_x\dfrac{dx}{dt} + f_y\dfrac{dy}{dt}\right)\mathbf{k}$ (by the Chain Rule).

Therefore

$$K = \tfrac{1}{2}m\,|\mathbf{v}|^2 = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt}\right)^2\right]$$

$$= \frac{m}{2}\left[\left(1 + f_x^2\right)\left(\frac{dx}{dt}\right)^2 + 2f_xf_y\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \left(1 + f_y^2\right)\left(\frac{dy}{dt}\right)^2\right]$$

(b) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2x}{dt^2}\,\mathbf{i} + \dfrac{d^2y}{dt^2}\,\mathbf{j} + \left[f_{xx}\left(\dfrac{dx}{dt}\right)^2 + 2f_{xy}\,\dfrac{dx}{dt}\,\dfrac{dy}{dt} + f_{yy}\left(\dfrac{dy}{dt}\right)^2 + f_x\,\dfrac{d^2x}{dt^2} + f_y\,\dfrac{d^2y}{dt^2}\right]\mathbf{k}$

(c) If $z = x^2 + y^2$, where $x = t\cos t$ and $y = t\sin t$, then $z = f(x, y) = t^2$.

$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t^2\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v} = (\cos t - t\sin t)\,\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j} + 2t\,\mathbf{k}$,

$K = \dfrac{m}{2}[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2] = \dfrac{m}{2}(1 + t^2 + 4t^2) = \dfrac{m}{2}(1 + 5t^2)$, and

$\mathbf{a} = (-2\sin t - t\cos t)\,\mathbf{i} + (2\cos t - t\sin t)\,\mathbf{j} + 2\,\mathbf{k}$. Notice that it is easier not to use the formulas in (a) and (b).