16.9: The Divergence Thm.

NOTES: Positive orientation refers to the normals.
label of "flux"
withdrawal deadline: 12/9

Since it was soooo successful, let's try the same format as the last section.

recall: Green's Thm

$$\oint_C \underset{P\,dx + Q\,dy}{F \cdot dr} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

and one of its vector forms

$$\underbrace{\oint_C \vec{F} \cdot \vec{n}\, ds}_{\substack{\text{integral of} \\ \vec{F}\cdot\vec{n} \text{ around} \\ \text{a boundary}}} = \underbrace{\iint_D \operatorname{div} \vec{F}(x,y)\, dA}_{\substack{\text{integral of} \\ \operatorname{div}\vec{F} \text{ over} \\ \text{a region}}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\iint_S \vec{F} \cdot \vec{n}\, ds = \iiint_E \operatorname{div} \vec{F}(x,y,z)\, dV$$

$\boxed{\text{Div: Interp.}}$

is $\vec{F}$ gives the velocity of a fluid w/ constant density... $\operatorname{div}\vec{F}$ gives the mass of the fluid leaving the region per unit time per unit volume.

<u>Divergence Thm:</u> Let E be a simple solid region & let S be the boundary surface of E, given w/ positive orientation. Let $\vec{F}$ be a vector field whose component fcts have cont. partials on an open region that contains E.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

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ex1: Calculate the flux of $\vec{F} = \langle x^2 z^3, 2xyz^3, xz^4 \rangle$ across the surface of the box w/vertices $(\pm 1, \pm 2, \pm 3)$

$$\text{div } F = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$$

$$\text{Flux} = \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} 8xz^3 \, dx\, dy\, dz = 0$$

$$\left[4x^2z^3\right]_{-1}^{1}$$

ex2: Calculate the flux of $\vec{F} = \dfrac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$ across the surface S consisting of $z = \sqrt{1-x^2-y^2}$ for $x^2+y^2 \le 1$ on the xy plane.

$$\text{div } \vec{F} = \left(\sqrt{x^2+y^2+z^2}\right)^{-1} + x\cdot\left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}\cdot 2x + \ldots$$

$$= \left(\sqrt{x^2+y^2+z^2}\right)^{-1} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}} + \ldots$$

$$= 3\left(\sqrt{x^2+y^2+z^2}\right)^{-1} - \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \cancel{\frac{3(x^2+y^2+z^2) - 1}{\sqrt{x^2+y^2+z^2}}} \qquad \frac{3(x^2+y^2+z^2) - (x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$\ldots \cancel{\text{div } \frac{\vec{r}}{|\vec{r}|} = \frac{1}{|\vec{r}|} \text{div } \vec{r} = \frac{3}{|r|}}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}$$

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$$\Rightarrow \quad \text{Flux} = \iiint_E \frac{2}{\sqrt{x^2+y^2+z^2}}\, dV$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 \frac{2}{\rho}\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \Big[\rho^2\Big]_0^1 \sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} 2\pi \sin\phi \, d\phi$$

$$= \Big[-2\pi \cos\phi\Big]_0^{\pi/2}$$

$$= -2\pi(0-1)$$

$$= 2\pi$$

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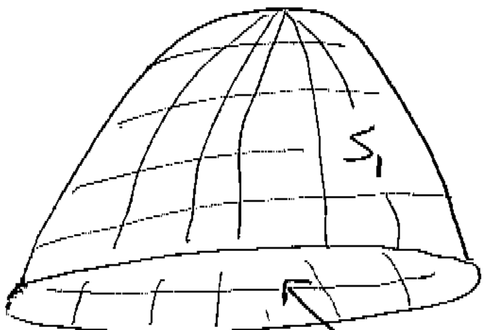
ex 3: recall from Test 3.

Find the flux of $\vec{F} = \langle -x, y, x^2+y^2 \rangle$
across $z = 9 - x^2 - y^2$ when $z \geq 0$.

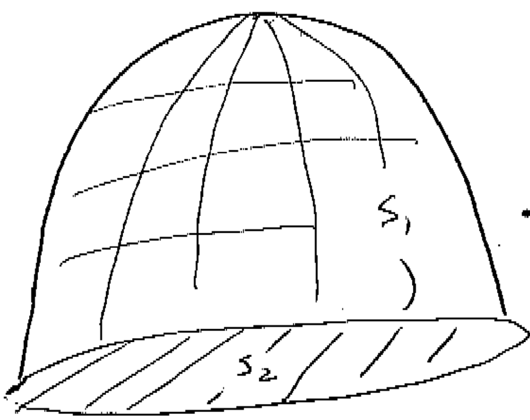


Good news.

$\operatorname{div}(\vec{F}) = -1 + 1 = 0$

Bad news

open @ bottom ... can't use the divergence Thm... directly.



$$\iint_{S_1 \cup S_2} \vec{F} \cdot d\vec{s} = \iint_{S_1} \vec{F} \cdot d\vec{s} + \iint_{S_2} \vec{F} \cdot ds$$

$$\Rightarrow \iint_{S_1} \vec{F} \cdot d\vec{s} = \iint_{S_1 \cup S_2} \vec{F} \cdot d\vec{s} - \iint_{S_2} \vec{F} \cdot d\vec{s}$$

$$= \iiint_E \underbrace{\operatorname{div} \vec{F}}_{0}\, dV - \iint_{S_2} \vec{F} \cdot d\vec{s}$$

$$= - \iint_{S_2} \vec{F} \cdot ds.$$

parametrize S_2.

$\vec{r}(R, \theta) = \langle R\cos\theta, R\sin\theta, 0 \rangle$

$\vec{r}_R = \langle \cos\theta, \sin\theta, 0 \rangle$

$\vec{r}_\theta = \langle -R\sin\theta, R\cos\theta, 0 \rangle$

$\vec{r}_R \times \vec{r}_\theta = \langle 0, 0, +R \rangle$

orientation

$$= - \iint_{D} \langle -R\cos\theta, R\sin\theta, R^2 \rangle \cdot \langle 0, 0, -R \rangle \, dA$$

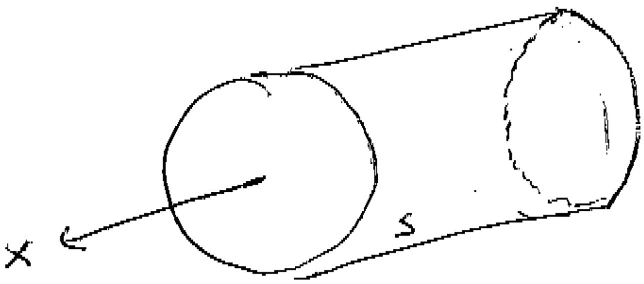
$$= + \int_0^3 \int_0^{2\pi} R^3 \, d\theta \, dR$$

$$= \left[\frac{2\pi R^4}{4} \right]_0^3 \qquad \frac{81\pi}{2}$$

$\boxed{\begin{matrix}16.9 \\ 5/5\end{matrix}}$

ex4: Find the ~~tot.~~ flux of $\vec{F} = \langle 3xy^2, xe^z, z^3 \rangle$ across the solid bounded by $y^2 + z^2 = 1$, $x = -1$, & $x = 2$.

$$\operatorname{div}(F) = 3y^2 + 3z^2$$



$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

$$= \iiint 3(y^2 + z^2)\, dv$$

$x = x$, $y = R\cos\theta$, $z = R\sin\theta$

$$= \int_0^1 \int_0^{2\pi} \int_{-1}^{2} 3R^2 \cdot R \, dx \, d\theta \, dR$$

$$= 18\pi \int_0^1 R^3 \, dR$$

$$= \frac{9}{2}\pi$$