16.4 : Green's Thm

Overview: Green's Thm establishes the link between line integrals and double integrals.

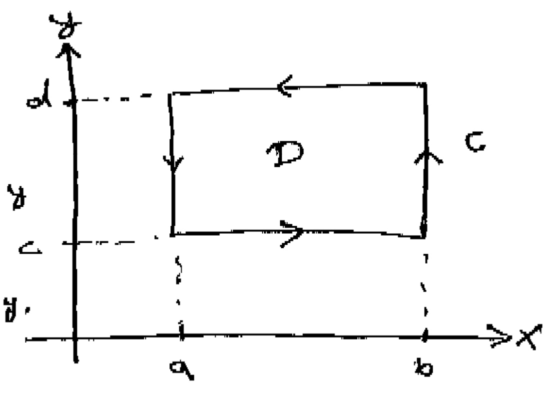
Recall our concluding thm from the preceding section...

Thm: Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open simply-connected region D. Suppose that P & Q have continuous 1st order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. Then $\vec{F}$ is conservative.

How far will this theorem take us?

Suppose $F = P(x,y)\vec{i} + Q(x,y)\vec{j}$

$$\oint \vec{F}\cdot dr = \oint_C P(x,y)\,dx + Q(x,y)\,dy$$



$$= \int_a^b P(x,c)\,dx + \int_c^d Q(b,y)\,dy + \int_b^a P(x,d)\,dx + \int_d^c Q(a,y)\,dy.$$

$$= \int_a^b P(x,c)\,dx - \int_a^b P(x,d)\,dx + \int_c^d Q(b,y)\,dy - \int_c^d Q(a,y)\,dy$$

$$= \int_a^b [P(x,c) - P(x,d)]\, dx + \int_c^d [Q(b,y) - Q(a,y)]\, dy.$$

reverse order

<u>scratch</u>

$$P(x,c) - P(x,d) = P(x,y)\Big|_{y=c}^{y=d} = \int_{y=c}^{y=d} \frac{\partial P}{\partial y}\, dy$$

$$Q(b,y) - Q(a,y) = Q(x,y)\Big|_{x=a}^{x=b} = \int_{x=a}^{x=b} \frac{\partial Q}{\partial x}\, dx$$

$$= -\int_a^b \int_c^d \frac{\partial P}{\partial y}\, dy + \int_c^d \int_a^b \frac{\partial Q}{\partial x}\, dx$$

$$= \int_a^b \int_c^d \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$$

so $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = \int_a^b \int_c^d \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\; dx\, dy$

where C is the positively oriented path around a <u>rectangular region.</u>

We will go into more detail as to why we can generalize after we have worked some examples... in the meantime.

<u>Green's Thm:</u> If C is a piecewise smooth, ~~curve~~ simple closed curve that bounds a region D, and if P & Q are continuous & have continuous 1st order partials along C and throughout D, then

what if $\vec{F} = \langle P, Q \rangle$ is conservative?

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

<u>Ex1:</u> Evaluate $I = \oint_C (3x-y)\,dx + (x+5y)\,dy$ around the unit circle $x = \cos t$; $y = \sin t$ on $0 \le t \le 2\pi$.

$dx = -\sin t\,dt$; $dy = \cos t\,dt$

$$I = \int_0^{2\pi} -(3\cos t - \sin t)\sin t + (\cos t + 5\sin t)\sin t\,dt$$

$$= \int_0^{2\pi} 2\sin t\cos t + 1\,dt$$

$$= \sin^2 t + t\Big]_0^{2\pi}$$

$$= 2\pi$$

$P = 3x - y$ & $Q = x + 5y$

$\dfrac{\partial P}{\partial y} = -1$ and $\dfrac{\partial Q}{\partial x} = 1$

$$I = \iint_D (1-(-1))\,dA$$

$$= \iint_D 2\,dA$$

$= 2\pi$ (twice the area of D).

Ex 2: Evaluate $I = \oint_C 2y + \sqrt{1+x^5}\,dx + (5x - e^{y^2})\,dy$

around the circle $x^2 + y^2 = 4$

$$P = 2y + \sqrt{1+x^5} \implies \frac{\partial P}{\partial y} = 2$$

$$Q = 5x - e^{y^2} \implies \frac{\partial Q}{\partial x} = 5$$

$$\text{so} \quad I = \iint_D (5-2)\,dA$$

$$= 3 \cdot \pi (2)^2 \quad \text{(3 times the area of D... D is enclosed by C)}$$

$$= 12\pi$$

Ex 3: ~~Evaluate~~ Interpret $I = \frac{1}{2}\oint_C -y\,dx + x\,dy$ geometrically.

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_D (1-(-1))\,dA$$

$$= \iint_D 1\,dA$$

so, the Area of the region D enclosed by C is:

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy = \iint_D 1\,dA$$

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Ex4: Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

If $x = a\cos t$ & $y = b\sin t$

$\Rightarrow dx = -a\sin t\,dt$ & $dy = b\cos t\,dt$

And $\cancel{A = \frac{1}{2}\oint_C b\cos t \cdot b\cos t\,dt = a}$

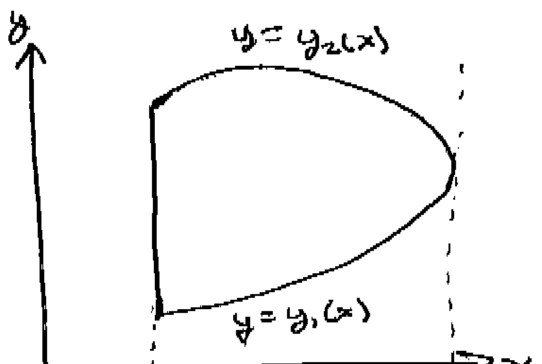
$$A = \frac{1}{2}\int_0^{2\pi} -b\sin t\,(-a\sin t)\,dt + a\cos t \cdot b\cos t\,dt$$

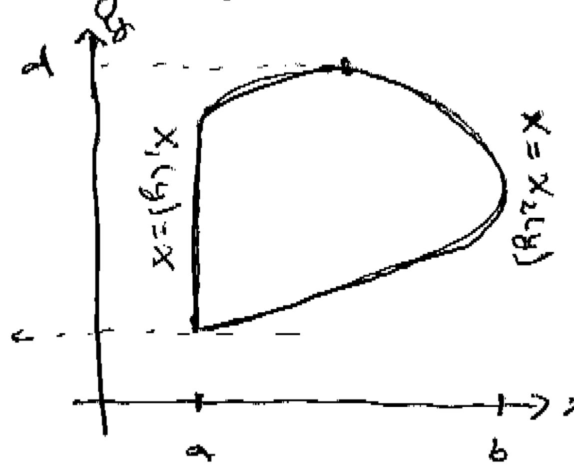
$$= \frac{1}{2}\int_0^{2\pi} ab\,dt$$

$$= \pi \cdot ab$$

Note: Green's Thm only holds if C is traversed once w/ CCW ~~ortn~~ or positive orientation.

Type I, II & simple regions.





regions that are both Type I & II are simple regions

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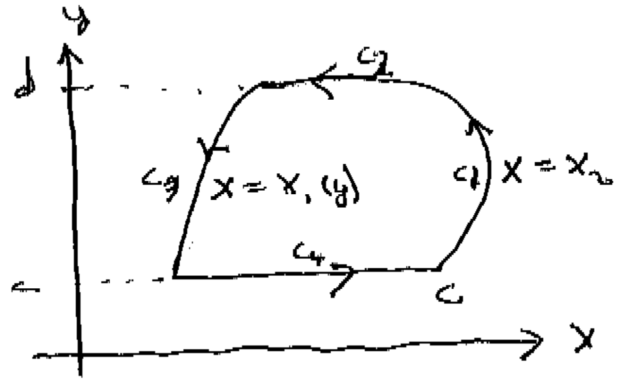
Sketch of the proof of Green's Thm over simple regions.



$I = \oint P\,dx + Q\,dy$

$$= \oint_C P(x,y)\,dx + \underbrace{\oint_C Q(x,y)\,dy}$$

$$\oint_{C_1} Q\,dy + \underbrace{\int_{C_2} Q\,dy}_{0} + \int_{C_3} Q\,dy + \underbrace{\int_{C_4} Q\,dy}_{0}$$

$$= \int_c^d Q(x_2(y), y)\,dy + \int_d^c Q(x_1(y), y)\,dy$$

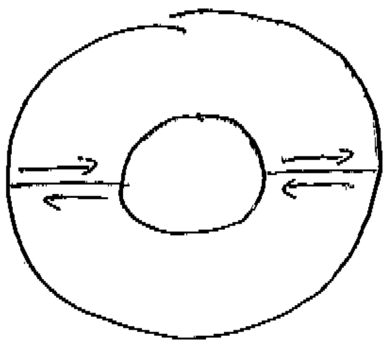
$$= \int_c^d \underbrace{\left[Q(x_2(y), y) - Q(x_1(y), y)\right]}_{Q(x,y)\Big]_{x=x_1(y)}^{x=x_2(y)}}\,dy$$

$$= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy.$$

$$= \iint_D \frac{\partial Q}{\partial x}\,dA$$

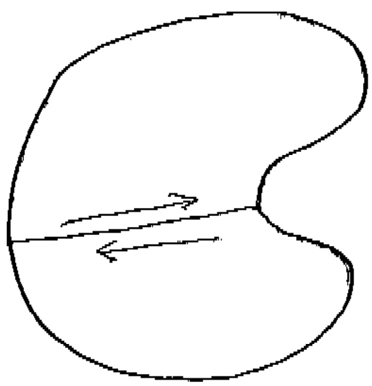
AND $\oint_C P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA.$

Why can we assume a simple region?



A donut is the union of two simple regions.

... but



a simple connected region