A.21 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS. JACOBIANS

Calculus w/ Analytic Geometry (2nd) by George Simmons. p 841 – 4

Our basic tools for integrating in polar, cylindrical, and spherical coordinates are the formulas

$$dA = r\,dr\,d\theta, \qquad dV = r\,dr\,d\theta\,dz, \qquad \text{and} \qquad dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta, \tag{1}$$

for the elements of area and volume in these three coordinate systems. However, the justifications we gave in Chapter 20 were purely intuitive and geometric. Our purpose in this brief final appendix is to describe a broader theoretical setting within which these formulas can be understood as merely different aspects of a single idea.

The problem that we now consider is the following: What happens to a multiple integral

$$\iint_R \cdots \int f(x, y, \ldots)\,dx\,dy \cdots$$

if we change the variables from $x, y, \ldots$ to $u, v, \ldots$?

We know the answer to this question in the case of a single variable: If $f(x)$ is continuous and the function $x = x(u)$ has a continuous derivative, then

$$\int_a^b f(x)\,dx = \int_c^d f[x(u)]\,\frac{dx}{du}\,du, \tag{2}$$

where $a = x(c)$ and $b = x(d)$. As an example of the use of this formula, we point out that the trigonometric substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$ enables us to write

$$\int_0^1 \sqrt{1 - x^2}\,dx = \int_0^{\pi/2} \cos\theta \cdot \cos\theta\,d\theta = \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\,d\theta = \frac{\pi}{4}.$$

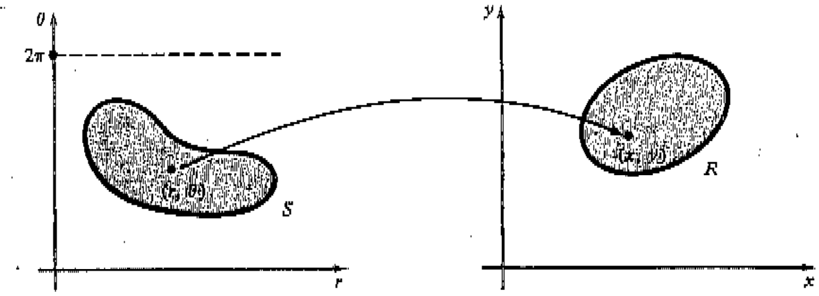

Figure A.20

Students should observe particularly that the change of variable in this calculation is accompanied by a corresponding change of the interval of integration.

Our only similar experience in the two-variable case is with changing double integrals from rectangular to polar coordinates by using the transformation equations

$$x = r \cos \theta, \qquad y = r \sin \theta. \tag{3}$$

Up to this stage we have interpreted these equations as expressing the rectangular coordinates of a given point in terms of its polar coordinates. However, they can also be interpreted as defining a *transformation* or *mapping* that carries points (r, θ) in the $r\theta$-plane over to points (x, y) in the xy-plane. That is, if a point (r, θ) is given, then equations (3) determine the corresponding point (x, y), as suggested in Fig. A.20. Further, in order to make this correspondence one-to-one, it is customary to restrict the point (r, θ) to lie in the part of the $r\theta$-plane specified by the inequalities $0 \leq r$, $0 \leq \theta < 2\pi$.

From this point of view, the formula for changing a double integral into polar coordinates [formula (3) in Section 20.4] can be written as

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r \cos \theta, r \sin \theta)\, r\, dr\, d\theta. \tag{4}$$

Thus, we are allowed to substitute $x = r \cos \theta$ and $y = r \sin \theta$ in the integral on the left, but we must then replace $dx\, dy$ by $r\, dr\, d\theta$ and R by the corresponding region S in the $r\theta$-plane. In our previous work we made no mention of the region S, but instead—and equivalently—changed the limits of integration on iterated integrals to describe the same region R in terms of polar coordinates.

Formula (4) is a special case of a very general formula for changing variables in double integrals. The detailed proof is beyond the scope of this book, but at least we can state the result. First we need a definition. Consider a pair of functions of two variables,

$$x = x(u, v), \qquad y = y(u, v), \tag{5}$$

and assume that they have continuous partial derivatives. The *Jacobian* of these functions is the determinant defined by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.^{*} \tag{6}$$

*Determinants of this form were first discussed by the German mathematician C. G. J. Jacobi (1804–1851). He did important work in the theory of elliptic functions, and applied his discoveries in astonishing ways to the theory of numbers. He also created a new and fruitful approach to theoretical dynamics. The Hamilton-Jacobi equations are part of the standard equipment of every student of mathematical physics. Also, Jacobi uttered the following magnificent and unforgettable defense of science for its own sake: "The sole aim of science is the honor of the human mind, and from this point of view a question about numbers is as important as a question about the system of the world."

This is often called a functional determinant, because it is a function of the variables u and v. As an example, we see that the Jacobian of the polar coordinates transformation (3) is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

The general change of variables formula for double integrals can now be stated as follows: *If (5) is a one-to-one transformation of a region S in the uv-plane onto a region R in the xy-plane, and if the Jacobian (6) is positive, then*

$$\iint_R f(x, y)\,dx\,dy = \iint_S f[x(u, v), y(u, v)]\,\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv. \tag{7}$$

Since r is the Jacobian of the polar coordinates transformation (3), it is clear that (4) is a special case of (7). Further, we can think of (7) as a two-dimensional extension of (2), with the derivative dx/du being replaced by the Jacobian $\partial(x, y)/\partial(u, v)$.

Formula (7) in turn can be extended to triple integrals. First, we define the *Jacobian* of the transformation

$$\begin{cases} x = x(u, v, w), \\ y = y(u, v, w), \\ z = z(u, v, w), \end{cases} \quad \text{by} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Then, under the similar assumptions, we have the following extension of (7):

$$\iiint_R f(x, y, z)\,dx\,dy\,dz = \iiint_S F(u, v, w)\,\frac{\partial(x, y, z)}{\partial(u, v, w)}\,du\,dv\,dw, \tag{8}$$

where $F(u, v, w) = f[x(u, v, w), y(u, v, w), z(u, v, w)]$. The main thing to notice here is that

$$dx\,dy\,dz \quad \text{is replaced by} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)}\,du\,dv\,dw.$$

Two important special cases of (8) are those of *cylindrical coordinates*,

$$\iiint_R f(x, y, z)\,dx\,dy\,dz = \iiint_S F(r, \theta, z)r\,dr\,d\theta\,dz,$$

where $F(r, \theta, z) = f(r\cos\theta, r\sin\theta, z)$; and *spherical coordinates*,

$$\iiint_R f(x, y, z)\,dx\,dy\,dz = \iiint_S F(\rho, \phi, \theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta,$$

where $F(\rho, \phi, \theta) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$. We leave it to the student to verify the spherical coordinates formula by using the transformation equations

$$\begin{cases} x = \rho\sin\phi\cos\theta, \\ y = \rho\sin\phi\sin\theta, \\ z = \rho\cos\phi, \end{cases}$$

to calculate the Jacobian

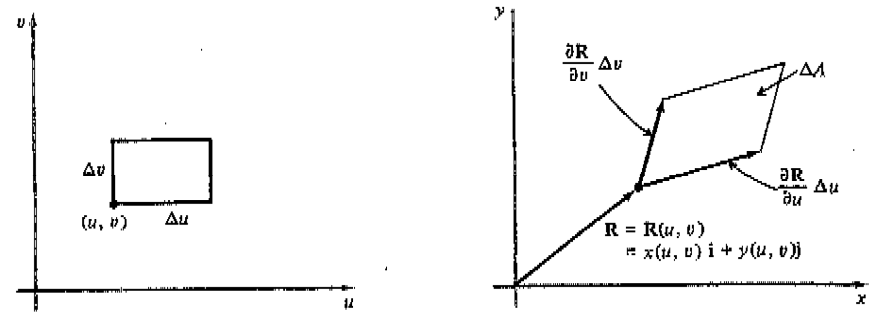

Figure A.21

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi.$$

It is in this way that we can understand a little more fully what lies behind formulas (1).

One question remains, and for the sake of simplicity we state it only for the two-variable case: What is the underlying reason for the presence of the Jacobian on the right side of formula (7)? We now give a very brief intuitive explanation of this by means of vectors. In the uv-plane the equations $u = a$ constant and $v = a$ constant determine a network of straight lines parallel to the axes, whereas in the xy-plane these equations determine a network of intersecting curves. A small rectangle in the uv-plane with sides Δu and Δv corresponds to a small parallelogram in the xy-plane (see Fig. A.21) with sides that can be written in vector form as

$$\frac{\partial \mathbf{R}}{\partial u}\,\Delta u = \left(\frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j}\right)\Delta u$$

and

$$\frac{\partial \mathbf{R}}{\partial v}\,\Delta v = \left(\frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j}\right)\Delta v,$$

approximately. In calculating the integral on the left side of (7) as a limit of sums, it is natural to abandon the usual rectangular cells and instead use these small parallelograms. If we denote by ΔA the area of the parallelogram in the figure, then ΔA equals the magnitude of the cross product of the two vectors given above. Since this cross product is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix}\Delta u\,\Delta v = \left[\frac{\partial(x, y)}{\partial(u, v)}\,\Delta u\,\Delta v\right]\mathbf{k},$$

we have

$$\Delta A = \frac{\partial(x, y)}{\partial(u, v)}\,\Delta u\,\Delta v. \tag{9}$$

This shows that the Jacobian plays the role of a local magnification factor for areas. Further, these remarks constitute a sketch of a proof of (7), because all that remains to establish (7) is to form the integral on the left side as a limit of sums and make use of (9).