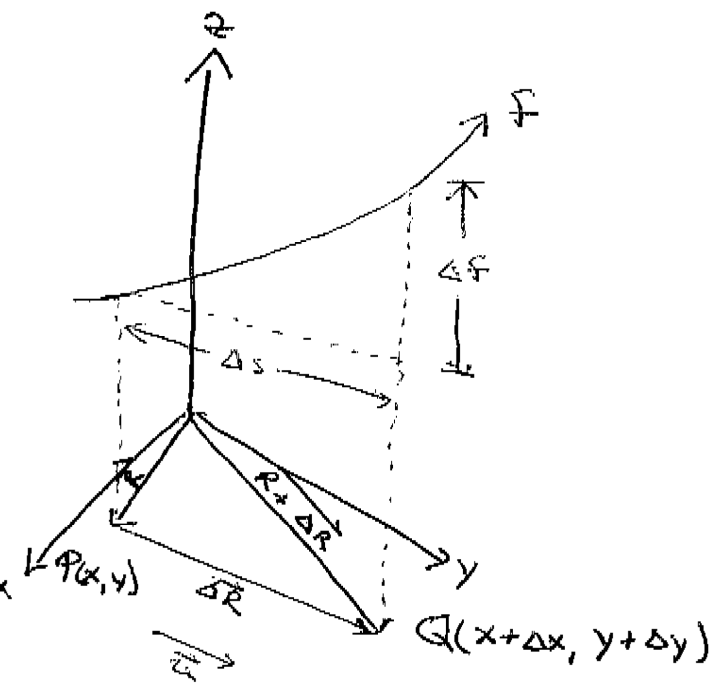
14.6: Directional Derivative & Gradient.

If $z = f(x,y)$ is a fct defined over $\mathbb{R}^2$, find the derivative of f @ a point $P(x,y)$ in the direction of $\vec{u}$ (a unit vector).



$\frac{df}{ds} = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s}$ is the derivative of f @ P in the direction of $\vec{u}$.

To calculate...

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \text{ where}$$

$\varepsilon_1, \varepsilon_2 \mapsto 0$ as $\Delta x, \Delta y \mapsto 0$. AND $\Delta x, \Delta y \mapsto 0$ as $\Delta s \mapsto 0$ (see pic).

$$\Rightarrow \frac{\Delta f}{\Delta s} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta s} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta s} + \varepsilon_1 \frac{\Delta x}{\Delta s} + \varepsilon_2 \frac{\Delta y}{\Delta s}$$

AND if $\Delta s \mapsto 0$

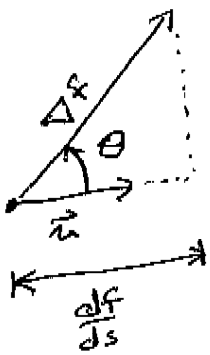
$$\Rightarrow \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} \quad (\partial\text{'s } \& \ d\text{'s})$$

This is like a special chain rule I where the independent variable s gives the distance from P to Q.

This can be written as

$$\frac{df}{ds} = \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}\cdot\underbrace{\left\langle \frac{dx}{ds}, \frac{dy}{ds} \right\rangle}$$

$$= \underbrace{\nabla f(x,y)}_{\text{Gradient vector}} \cdot \underbrace{\frac{d\vec{R}}{ds}}_{\text{unit vector in the same direction as } \vec{u}.}$$



$$= \nabla f \cdot \vec{u} \qquad \text{(alg. formula)}$$

$$= |\nabla f| \cos\theta \qquad \text{(geo. formula)}$$

While the derivation focused on the directional derivative, the main point is the gradient ∇f.

<u>∇ Property 1</u>: The directional derivative $\frac{df}{ds}$ in any direction is ~~of~~ the scalar projection of ∇f in that direction.

<u>∇ Property 2</u>: The vector ∇f points in the direction f increases most rapidly.

WHY?

<u>∇ Property 3</u>: The length of ∇f is the max rate of increase of f.

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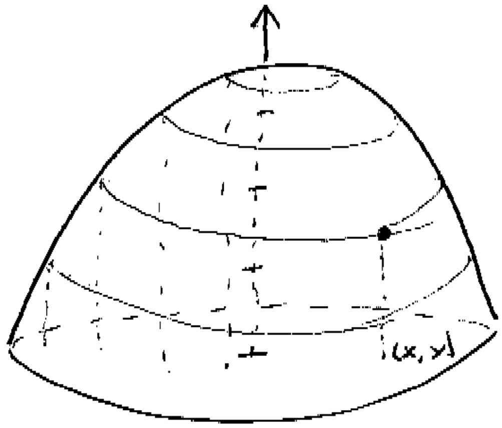
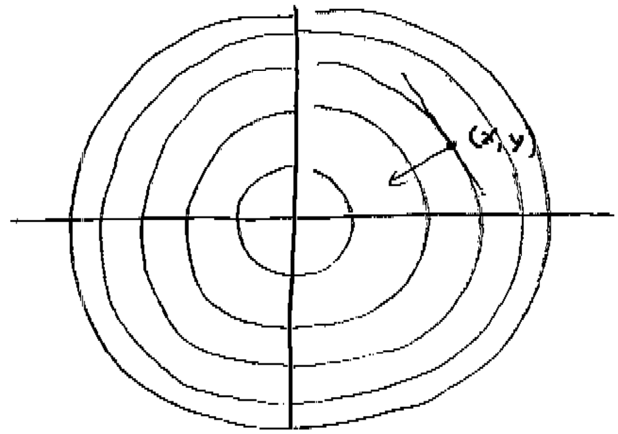
Making connections





In what direction(s) would the directional derivative be zero? Along the level curve (Tangent)

In what direction would the directional derivative be maximized (∇f)? Normal to the level curve (Normal to tangent).

Does this make sense?

Topomaps ... (Fig 12).

How do we visualize $f(x, y, z) = w$ in $\mathbb{R}^4$?
If $k = w$ is const, we have a level surface $f(x, y, z) = k$.

What does the gradient represent by analogy w/the example in $\mathbb{R}^3$?

Ex: Find the eqts of
(a) the tangent plane
(b) the normal line
to $x - z = 4\arctan(yz)$ @ $P(1+\pi, 1, 1)$

$$\Rightarrow \quad 0 = z - x + 4\arctan(yz)$$

and
$$F_x = -1 \Big|_P \quad -1$$
$$F_y = \frac{z}{1+(yz)^2}\Big|_P \quad \frac{1}{2}$$
$$F_z = 1 + \frac{y}{1+(yz)^2}\Big|_P \quad \frac{3}{2}$$
$$\nabla f(1+\pi, 1, 1) = \left\langle -1, \frac{1}{2}, \frac{3}{2} \right\rangle$$
normal to the surface @ point P.

(a) Plane

$$-1(x-(1+\pi)) + \frac{1}{2}(y-1) + \frac{3}{2}(z-1) = 0$$

(b) The normal line

$$\frac{x-(1+\pi)}{-1} = \frac{y-1}{1/2} = \frac{z-1}{3/2}$$