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14.5: The chain Rule.

recall, if $w(t) = f(x(t))$, then

$$w'(t) = f'(x(t)) \cdot x'(t)$$

$$\text{or } \frac{df}{dx} \cdot \frac{dx}{dt}$$

Now, if $w(t) = f(x(t), y(t))$ then we need case I of the chain rule.

The Chain Rule I: suppose $z = f(x,y)$ is a differentiable fct of x & y where $x = g(t)$ & $y = h(t)$ are both differentiable fcts of t. Then z is a differentiable fct of t &

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\text{or } \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \quad (z \text{ instead of } f)$$

Ex 1: If $w = 3x^2 + 2xy - y^2$ where $x = \cos t$ & $y = \sin t$, find $\frac{dw}{dt}$.

$$\frac{dw}{dt} = (6x + 2y)(-\sin t) + (2x - 2y)\cos t$$

$$= (6\cos t + 2\sin t)(-\sin t) + (2\cos t - 2\sin t)\cos t$$

$$\cancel{= -4\cos t \sin t - 2(\sin^2 t + \cos^2 t)}$$

$$= -8\cos t \sin t - 2\sin^2 t + 2\cos^2 t.$$

CHECK BY SUBING x & y IN FIRST.

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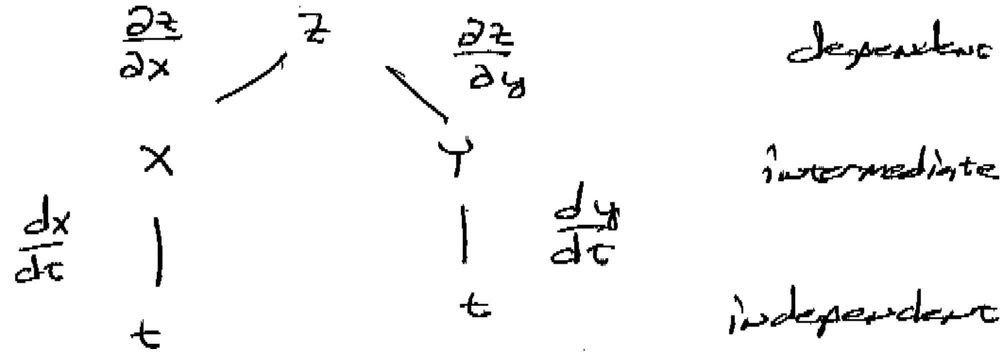
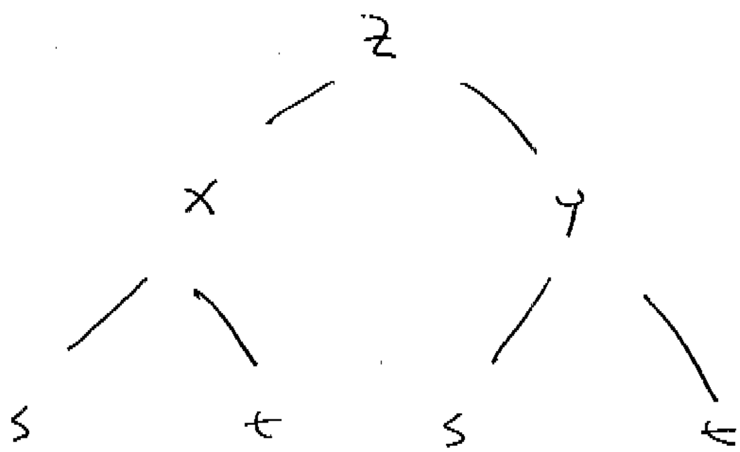
VOCAB: we call x & y intermediate variables & the independent variable.



The Chain Rule (II): Suppose that $z = f(x, y)$ is a diff. fct. of x & y where $x = g(s,t)$ & $y = h(s,t)$ are diff. fcts of s & t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad \& \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$



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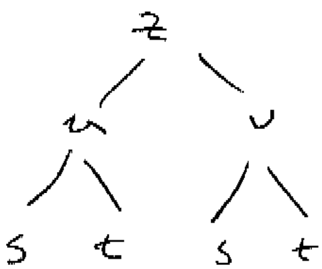
<u>ex2</u>: Find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$ if $z = \cot\left(\frac{v}{u}\right)$,

$u = 2s - 3t$ & $v = 5s + t$



$$\frac{\partial z}{\partial u} = -\csc^2\left(\frac{v}{u}\right) \cdot -\frac{v}{u^2}$$

$$\frac{\partial z}{\partial v} = -\csc^2\left(\frac{v}{u}\right) \cdot \frac{1}{u}$$

so $$\frac{\partial z}{\partial s} = \frac{v}{u^2}\csc^2\left(\frac{v}{u}\right) \cdot 2 + -\frac{1}{u}\csc^2\left(\frac{v}{u}\right) \cdot 5$$

$$= \frac{2(5s+t)}{(2s-3t)^2}\csc^2\left(\frac{5s+t}{2s-3t}\right) - \frac{5}{2s-3t}\csc^2\left(\frac{5s+t}{2s-3t}\right)$$

$\frac{\partial z}{\partial t}$ can be found as a HW exercise.

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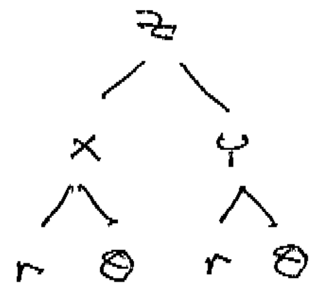
Down the road, we will be learning about cylindrical coordinates which will require that we rewrite x & y in terms of r & θ.

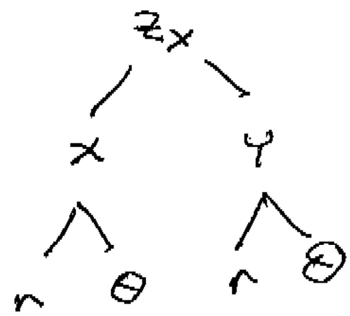
Ex 3: Assuming cont. deriv's, if $z = f(x,y)$ where $x = r\cos\theta$ & $y = r\sin\theta$, find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}, \frac{\partial^2 z}{\partial r \partial \theta}$



a) $z_r = z_x \cos\theta + z_y \sin\theta$

b) $z_\theta = -z_x \cdot r\sin\theta + z_y \cdot r\cos\theta$

c) $\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial \theta}\right) = z_{\theta r}$



Note, this will require two product rules and two chain rules $\frac{\partial}{\partial r}(z_x)$ & $\frac{\partial}{\partial r}(z_y)$

$$z_{\theta r} = \left[(z_{xx} x_r + z_{xy} y_r)(-r\sin\theta) + z_x(-\sin\theta)\right] + \left[(z_{yx} x_r + z_{yy} y_r)(r\cos\theta) + z_y \cos\theta\right]$$

$$= -r\sin\theta\cos\theta\, z_{xx} - r\sin^2\theta\, z_{xy} - \sin\theta\, z_x + r\cos^2\theta\, z_{yx} + r\sin\theta\cos\theta\, z_{yy} + \cos\theta\, z_y.$$

$$= \cos\theta\, z_y - \sin z_x + r\sin\theta\cos\theta(z_{yy} - z_{xx}) + r(\cos^2\theta - \sin^2\theta) z_{yx}$$

since $z_{xy} = z_{yx}$ by Clairaut's Thm.

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suppose an eq^t of the form
$F(x, y) = 0$ defines y implicitly
as a fct of x.

That is, $y = f(x)$ where $F(x, f(x)) = 0$
$\forall x \in D_f$. If F is ~~dif~~ diff, then
by case I of the chain rule...

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial F}{\partial y} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

<u>Ex4</u>: Find $\frac{dy}{dx}$ if $y^5 + x^2 y^3 = 1 + ye^{x^2}$