14.4: Tangent Planes & Linear Approximations

Reading assignment.

As tangent lines approx curves, so tangent planes approx surfaces. Thus, we can use a tangent plane as the linear approximation to a surface near a point (assuming it exists)

use graph to explain

Ex 1: Find the tangent plane to $z = 2xy^3 - 5x^2$ @ $(3,2,3)$ & use it to approx the surface when $x = 3.1$ & $y = 1.95$.

$$z_x = 2y^3 - 10x \Big|_{(3,2)} -14$$

$$z_y = 6xy^2 \Big|_{(3,2)} 72$$

$\Rightarrow T(x,y) - 3 = -14(x-3) + 72(y-2)$

now, approx the surface @ $(3.1, 2.9)$

$$T(3.1, 2.9) = 3 - 14(0.1) + 72(-0.05) = -2$$

comparing to $z\Big|_{(3.1,2.9)} \approx -2.078$ which seems close enough to the approx to deem linear approximation valuable.

use graph to explain

It doesn't always work this well. Consider

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{else} \\ 0, & \text{origin} \end{cases}$$ w/graph in fig 4 on p 894.

w/o proof (which would require 14.2) we have that $f_x(0,0) = f_y(0,0) = 0$ (the partials exist). Yet, neither is cont @ the origin. Further, the lin. approx would be $f(x,y) \approx 0$, yet $f(x,y) = \frac{1}{2}$ @ all pts other than the origin on $y = x$.

The previous example showed when a linear approx (tangent plane) is not a good approx of a fct.

We say that differentiable fcts can be approximated by their tangent plane. So, what does "differentiable" mean in this context.

Dfn: If $z = f(x,y)$, then f is differentiable @ (a,b) if Δz can be written as

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 & $\varepsilon_2 \mapsto 0$ as $(\Delta x, \Delta y) \mapsto (0,0)$.

It is easier to work w/

Thm: If the partials f_x & f_y exist near (a,b) and are cont @ (a,b), then f is diff @ (a,b).

Ex2: Show that $f(x,y) = 2xy^3 - 5x^2$ is differentiable @ $(3,2)$ and find its linear approximation. Then, use it to approx $f(3.1, 1.95)$

$$f_x(x,y) = 2y^3 - 10x \Big|_{(3,2)} -14$$

$$f_y(x,y) = 6xy^2 \Big|_{(3,2)} 72$$

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↳ f_x & f_y exist around $(3,2)$ and are cont @ $(3,2)$ — polynomials — so f is diff. @ $(3,2)$.

the linearization is

$$L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$$

$$= 3 - 14(x-3) + 72(y-2)$$

$L(3.1, 1.95) = -2$ which is reasonably close to $f(3.1, 1.95) \simeq -2.078$.

<u>Differentials</u>

Recall from 3.10 that if $y = f(x)$, we defined dx to be an independent variable & $dy = f'(x)dx$.

The parallel to $z = f(x,y)$ is straightforward

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

$$= \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

this is also called the <u>total differential</u>

How does this relate to the tangent plane?

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If we let $\Delta x = x - a = dx$
$\Delta y = y - b = dy$

then $L(x,y) = f(x,y) \approx f(a,b) + \underbrace{f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{dz}$

so $f(x,y) \approx f(a,b) + dz$

Ex 3: The length & width of a rectangle are measured @ 18 in & 27 in respectively w/ an error in measurement of at most 0.1 in in length & 0.05 in in width. Use differentials to estimate the max error in the calculated area.

$$A(x,y) = A(L,w)$$
$$= L \cdot w$$

$$A_L = w \quad \& \quad A_w = L$$

$$dA = w \cdot dL + L \cdot dw$$

$$\text{max} = 27 \cdot (.1) + 18(.05)$$
$$= 3.6 \text{ in}^2$$

Note on the differential w/ 3 or more variables —— p 898 error in text where it misses the pt that dz is an estimate.