4.2 Linear Programming: Graphical Methods

OBJECTIVE

- To use graphical methods to find the optimal value of a linear function subject to constraints

Application Preview

Many practical problems in business and economics involve complex relationships among capital, raw materials, labor, and so forth. Consider the following example.

A farm co-op has 6000 acres available to plant with corn and soybeans. Each acre of corn requires 9 gallons of fertilizer/herbicide and 3/4 hour of labor to harvest. Each acre of soybeans requires 3 gallons of fertilizer/herbicide and 1 hour of labor to harvest. The co-op has available at most 40,500 gallons of fertilizer/herbicide and at most 5250 hours of labor for harvesting. If the profits per acre are \$60 for corn and \$40 for soybeans, how many acres of each crop should the co-op plant in order to maximize their profit? What is the maximum profit? (See Example 1.)

The farm co-op's maximum profit can be found by using a mathematical technique called **linear programming.** Linear programming can be used to solve problems such as this if the limits on the variables (called **constraints**) can be expressed as linear inequalities and if the function that is to be maximized or minimized (called the **objective function**) is a linear function.

Feasible Regions and Solutions

Linear programming is widely used by businesses for problems that involve many variables (sometimes more than 100). In this section we begin our study of this important technique by considering problems involving two variables. With two variables we can use graphical methods to help solve the problem. The constraints form a system of linear inequalities in two variables that we can solve by graphing. The solution of the system of constraint inequalities determines a region, any point of which may yield the *optimal* (maximum or minimum) value for the objective function.* Hence any point in the region determined by the constraints is called a **feasible solution,** and the region itself is called the **feasible region.** In a linear programming problem, we seek the feasible solution that maximizes (or minimizes) the objective function.

EXAMPLE 1 *Profit Maximization (Application Preview)*

A farm co-op has 6000 acres available to plant with corn and soybeans. Each acre of corn requires 9 gallons of fertilizer/herbicide and 3/4 hour of labor to harvest. Each acre of soybeans requires 3 gallons of fertilizer/herbicide and 1 hour of labor to harvest. The co-op has available at most 40,500 gallons of fertilizer/herbicide and at most 5250 hours of labor for harvesting. If the profits per acre are \$60 for corn and \$40 for soybeans, how many acres of each crop should the co-op plant in order to maximize their profit? What is the maximum profit?

Solution

Recall from Example 4 in Section 4.1 that if x is the number of acres of corn and y is the number of acres of soybeans, then the constraints for this farm co-op application can be written as the following system of inequalities.

$$\begin{aligned} x + \quad y &\le \ \ 6{,}000 \\ 9x + 3y &\le 40{,}500 \\ \frac{3}{4}x + \quad y &\le \ \ 5{,}250 \\ x \ge 0, y &\ge 0 \end{aligned}$$

*The region determined by the constraints must be *convex* for the optimal to exist. A convex region is one such that for any two points in the region, the segment joining those points lies entirely within the region. We restrict our discussion to convex regions.

Because the profit P is \$60 for each of the x acres of corn and \$40 for each of the y acres of soybeans, the function that describes the co-op's profit is given by

$$\text{Profit} = P = 60x + 40y \quad \text{(in dollars)}$$

Thus the linear programming problem for the co-op can be stated as follows.

$$\begin{aligned} &\text{Maximize Profit} \quad P = 60x + 40y \\ &\text{Subject to:} \quad x + y \le 6{,}000 \\ &\qquad 9x + 3y \le 40{,}500 \\ &\qquad \frac{3}{4}x + y \le 5{,}250 \\ &\qquad x \ge 0, y \ge 0 \end{aligned}$$

The solution of the system of inequalities, or constraints (found in Example 4 in Section 4.1), forms the feasible region shaded in Figure 4.9(a). *Any* point inside the shaded region or on its boundary is a feasible (possible) solution of the problem. For example, point A (1000, 2000) is in the feasible region, and at this point the profit is $P = 60(1000) + 40(2000) = 140{,}000$ dollars.

To find the maximum value of $P = 60x + 40y$, we cannot possibly evaluate P at every point in the feasible region. However, many points in the feasible region may correspond to the same value of P. For example, at point A in Figure 4.9(a), the value of P is 140,000 and the profit function becomes

$$60x + 40y = 140{,}000$$

or

$$y = \frac{140{,}000 - 60x}{40} = 3500 - \frac{3}{2}x$$

The graph of this function is a line with slope $m = -3/2$ and y-intercept 3500. Many points in the feasible region lie on this line (see Figure 4.9(b)), and their coordinates all result in a profit $P = 140{,}000$. Any point in the feasible region that results in profit P satisfies

$$P = 60x + 40y$$

or

$$y = \frac{P - 60x}{40} = \frac{P}{40} - \frac{3}{2}x$$

In this form we can see that different P-values change the y-intercept for the line but the slope is always $m = -3/2$, so the lines for different P-values are parallel. Figure 4.9(b) shows the feasible region and the lines representing the objective function for $P = 140{,}000$, $P = 315{,}000$, and $P = 440{,}000$. Note that the line corresponding to $P = 315{,}000$ intersects the feasible region at the point (3750, 2250). Values of P less than 315,000 give lines that pass through the feasible region, but represent a profit less than \$315,000 (such as the line for $P = 140{,}000$ through point A). Similarly, values of P greater than 315,000 give lines that miss the feasible region, and hence cannot be solutions of the problem.

Thus, the farm co-op's maximum profit, subject to the constraints (i.e., the solution of the co-op's linear programming problem), is $P = \$315{,}000$ when $x = 3750$ and $y = 2250$. That is, when the co-op plants 3750 acres of corn and 2250 acres of soybeans, it achieves maximum profit of \$315,000.

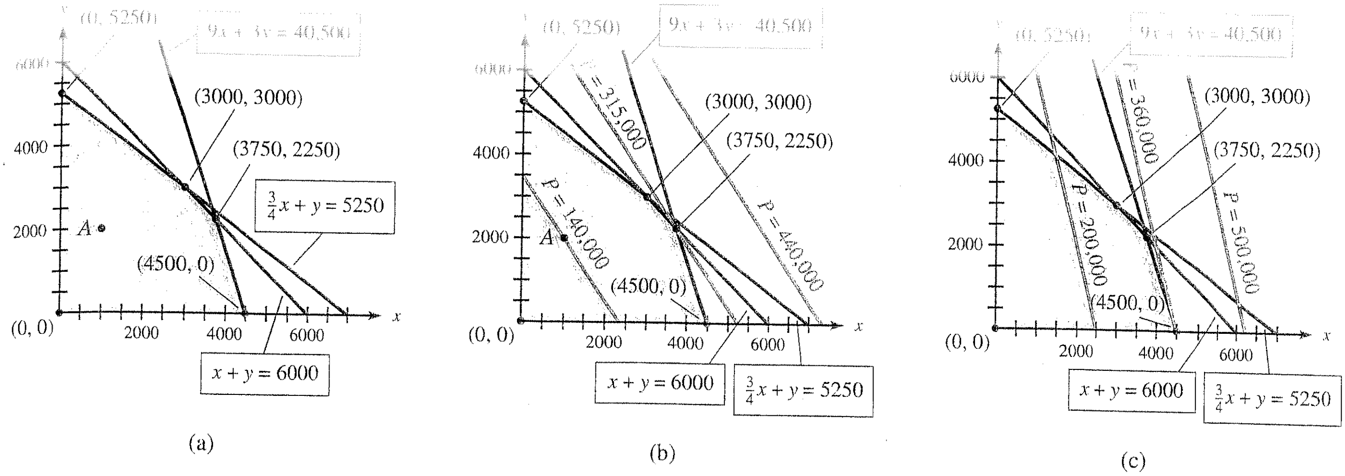


Figure 4.9

Notice in Example 1 that the objective function was maximized at one of the "corners" (vertices) of the feasible region. This is not a coincidence; it turns out that a corner will always lie on the profit line corresponding to the maximum profit.

Suppose in Example 1 that the objective had been to maximize $P = 80x + 20y$ over the same constraint region. Then the graphs of $P = 80x + 20y$, or $y = \frac{P}{20} - 4x$, for $P = 200{,}000$, $P = 360{,}000$, and $P = 500{,}000$ would result in the lines shown in Figure 4.9(c). Notice that the corner (4500, 0) is the only feasible point where $P = 360{,}000$. This figure also shows that any P-value greater than 360,000 will result in a line that misses the feasible region and that any other point inside the feasible region corresponds to a P-value less than 360,000. Thus the maximum value is $P = 360{,}000$ and it occurs at the corner (4500, 0). Note in both cases the objective function was maximized at one of the "corners" of the feasible region.

Solving Graphically

The feasible region in Figure 4.9 is an example of a **closed and bounded region** because it is entirely enclosed by, and includes, the lines associated with the constraints.

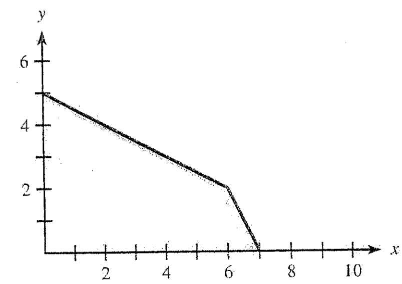
Solutions of Linear Programming Problems

1. When the feasible region for a linear programming problem is closed and bounded, the objective function has a maximum value and a minimum value.
2. When the feasible region is not closed and bounded, the objective function may have a maximum only, a minimum only, or no solution.
3. If a linear programming problem has a solution, then the optimal (maximum or minimum) value of an objective function occurs at a corner of the feasible region determined by the constraints.
4. If the objective function has its optimal value at two corners, then it also has that optimal value at any point on the line (boundary) connecting those two corners.

Thus, for a closed and bounded region, we can find the maximum or minimum value of the objective function by evaluating the function at each of the corners of the feasible region formed by the solution of the constraint inequalities. If the feasible region is not closed and bounded, we must check to make sure the objective function has an optimal value.

The steps involved in solving a linear programming problem are as follows.

Linear Programming (Graphical Method)

Procedure	Example
To find the optimal value of a function subject to constraints:	Find the maximum and minimum values of $C = 2x + 3y$ subject to the constraints $\begin{cases} x + 2y \le 10 \\ 2x + y \le 14 \\ x \ge 0, y \ge 0 \end{cases}$
1. Write the objective function and constraint inequalities from the problem.	1. Objective function: $C = 2x + 3y$ Constraints: $x + 2y \le 10$ $2x + y \le 14$ $x \ge 0, y \ge 0$
2. Graph the solution of the constraint system. (a) If the feasible region is closed and bounded, proceed to Step 3. (b) If the region is not closed and bounded, check whether an optimal value exists. If not, state this. If so, proceed to Step 3.	2. The constraint region is closed and bounded. See Figure 4.10.  Figure 4.10
3. Find the corners of the resulting feasible region. This may require simultaneous solution of two or more pairs of boundary equations.	3. Corners are (0, 0), (0, 5), (6, 2), (7, 0).
4. Evaluate the objective function at each corner of the feasible region determined by the constraints.	4. At (0, 0), $C = 2x + 3y = 2(0) + 3(0) = 0$ At (0, 5), $C = 2x + 3y = 2(0) + 3(5) = 15$ At (6, 2), $C = 2x + 3y = 2(6) + 3(2) = 18$ At (7, 0), $C = 2x + 3y = 2(7) + 3(0) = 14$
5. If two corners give the optimal value of the objective function, then all points on the boundary line joining these two corners also optimize the function.	5. The function is maximized at $x = 6, y = 2$. The maximum value is $C = 18$. The function is minimized at $x = 0, y = 0$. The minimum value is $C = 0$.

EXAMPLE 2 ***Maximizing Revenue***

Chairco manufactures two types of chairs, standard and plush. Standard chairs require 2 hours to construct and finish, and plush chairs require 3 hours to construct and finish. Upholstering takes 1 hour for standard chairs and 3 hours for plush chairs. There are 240 hours per day available for construction and finishing, and 150 hours per day are available for upholstering. If the revenue for standard chairs is \$89 and for plush chairs is \$133.50, how many of each type should be produced each day to maximize revenue?

Solution

Let x be the number of standard chairs produced each day, and let y be the number of plush chairs produced. Then the daily revenue function is given by $R = 89x + 133.5y$. There are constraints for construction and finishing (no more than 240 hours/day) and for upholstering (no more than 150 hours/day). Thus we have the following.

$$\text{Construction/finishing constraint:} \quad 2x + 3y \leq 240$$
$$\text{Upholstering constraint:} \quad x + 3y \leq 150$$

Because all quantities must be nonnegative, we also have the constraints $x \geq 0$ and $y \geq 0$.

Thus we seek to solve the following problem.

$$\text{Maximize } R = 89x + 133.5y \text{ subject to}$$
$$\begin{cases} 2x + 3y \leq 240 \\ x + 3y \leq 150 \\ x \geq 0, y \geq 0 \end{cases}$$

The feasible set is the closed and bounded region shaded in Figure 4.11. The corners of the feasible region are (0, 0), (120, 0), (0, 50), and (90, 20). All of these are obvious except (90, 20), which can be found by solving $2x + 3y = 240$ and $x + 3y = 150$ simultaneously. Testing the objective function at the corners gives the following.

At (0, 0), $R = 89x + 133.5y = 89(0) + 133.5(0) = 0$

At (120, 0), $R = 89(120) + 133.5(0) = 10{,}680$

At (0, 50), $R = 89(0) + 133.5(50) = 6675$

At (90, 20), $R = 89(90) + 133.5(20) = 10{,}680$

Maximum at two corners

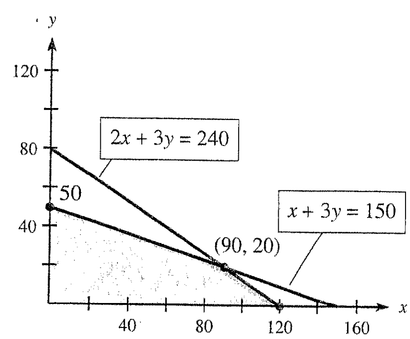


Figure 4.11

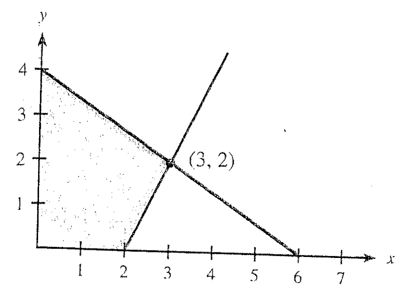
Thus the maximum revenue of \$10,680 occurs at either the point (120, 0) or the point (90, 20). This means that the revenue function will be maximized not only at these two corner points but also at any point on the segment joining them. Since the number of each type of chair must be an integer, Chairco has maximum revenue of \$10,680 at any pair of integer values along the segment joining (120, 0) and (90, 20). For example, the point (105, 10) is on this segment, and the revenue at this point is also \$10,680:

$$89x + 133.5y = 89(105) + 133.5(10) = 10{,}680$$

Checkpoint

1. Find the maximum and minimum values of the objective function $f = 4x + 3y$ on the shaded region in the figure, determined by the following constraints.

$$\begin{cases} 2x + 3y \le 12 \\ 4x - 2y \le 8 \\ x \ge 0, y \ge 0 \end{cases}$$



Although the examples so far have all involved closed and bounded regions, similar procedures apply for an unbounded region, although optimal solutions are no longer guaranteed.

EXAMPLE 3 *Minimization*

If possible, find the maximum and minimum values of $C = x + y$ subject to the constraints

$$\begin{cases} 3x + 2y \ge 12 \\ x + 3y \ge 11 \\ x \ge 0, y \ge 0 \end{cases}$$

Solution

The graph of the constraint system is shown in Figure 4.12(a). Note that the feasible region is not closed and bounded, so we must check whether optimal values exist. This check is done by graphing $C = x + y$ for selected values of C and noting the trend. Figure 4.12(b) shows the solution region with graphs of $C = x + y$ for $C = 3$, $C = 5$, and $C = 8$. Note that the objective function has a minimum but no maximum.

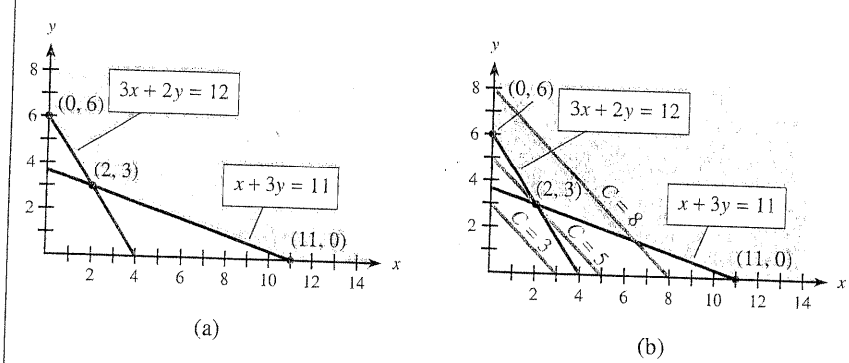


Figure 4.12

The corners (0, 6) and (11, 0) can be identified from the graph. The third corner, (2, 3), can be found by solving the equations $3x + 2y = 12$ and $x + 3y = 11$ simultaneously:

$$\begin{aligned} 3x + 2y &= 12 \\ -3x - 9y &= -33 \\ \hline -7y &= -21 \\ y &= 3 \\ x &= 2 \end{aligned}$$

Examining the value of C at each corner point, we have

$$\begin{aligned} \text{At } (0, 6), &\quad C = x + y = 6 \\ \text{At } (11, 0), &\quad C = x + y = 11 \\ \text{At } (2, 3), &\quad C = x + y = 5 \end{aligned}$$

Thus, we conclude the following:

Minimum value of $C = x + y$ is 5 at (2, 3).
Maximum value of $C = x + y$ does not exist.

Note that C can be made arbitrarily large in the feasible region.

Checkpoint

2. Find the maximum and minimum values (if they exist) of the objective function $g = 3x + 4y$ subject to the following constraints.

$$\begin{aligned} x + 2y \geq 12, &\quad x \geq 0 \\ 3x + 4y \geq 30, &\quad y \geq 2 \end{aligned}$$

EXAMPLE 4 *Minimizing Production Costs*

Two chemical plants, one at Macon and one at Jonesboro, produce three types of fertilizer, low phosphorus (LP), medium phosphorus (MP), and high phosphorus (HP). At each plant, the fertilizer is produced in a single production run, so the three types are produced in fixed proportions. The Macon plant produces 1 ton of LP, 2 tons of MP, and 3 tons of HP in a single operation, and it charges \$600 for what is produced in one operation, whereas one operation of the Jonesboro plant produces 1 ton of LP, 5 tons of MP, and 1 ton of HP, and it charges \$1000 for what it produces in one operation. If a customer needs 100 tons of LP, 260 tons of MP, and 180 tons of HP, how many production runs should be ordered from each plant to minimize costs?

Solution

If x represents the number of operations requested from the Macon plant and y represents the number of operations requested from the Jonesboro plant, then we seek to minimize cost

$$C = 600x + 1000y$$

The following table summarizes production capabilities and requirements.

	Macon Plant	Jonesboro Plant	Requirements
Units of LP	1	1	100
Units of MP	2	5	260
Units of HP	3	1	180

Using the number of operations requested and the fact that requirements must be met or exceeded, we can formulate the following constraints.

$$\begin{cases} x + y \geq 100 \\ 2x + 5y \geq 260 \\ 3x + y \geq 180 \\ x \geq 0, y \geq 0 \end{cases}$$

Graphing this system gives the feasible set shown in Figure 4.13. The objective function has a minimum even though the feasible set is not closed and bounded. The corners are (0, 180), (40, 60), (80, 20), and (130, 0), where (40, 60) is obtained by solving $x + y = 100$

and $3x + y = 180$ simultaneously, and where (80, 20) is obtained by solving $x + y = 100$ and $2x + 5y = 260$ simultaneously.

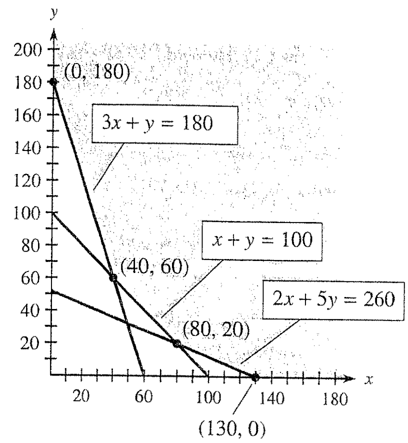


Figure 4.13

Evaluating $C = 600x + 1000y$ at each corner, we obtain

At (0, 180), $C = 180{,}000$
At (40, 60), $C = 84{,}000$
At (80, 20), $C = 68{,}000$
At (130, 0), $C = 78{,}000$

Thus, by placing orders requiring 80 production runs from the Macon plant and 20 production runs from the Jonesboro plant, the customer's needs will be satisfied at a minimum cost of $68,000.

EXAMPLE 5 *Maximization Subject to Constraints*

Use a graphing utility to maximize $f = 5x + 11y$ subject to the constraints

$$5x + 2y \leq 54$$
$$2x + 4y \leq 60$$
$$x \geq 0, y \geq 0$$

Solution

We write the inequalities above as equations, solved for y. Graphing these equations with a graphing calculator and using shading show the closed and bounded region satisfying the inequalities (see Figure 4.14). By using an INTERSECT command with pairs of lines that form the borders of this region, we see that the boundaries intersect at (0, 0), (0, 15), (6, 12), and (10.8, 0). These points can also be found algebraically. Testing the objective function at each of these corners gives the following values of f:

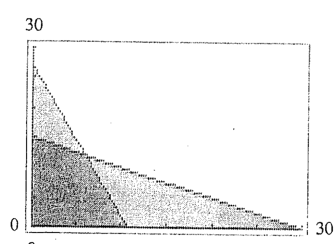


Figure 4.14

At (0, 0), $f = 5x + 11y = 0$
At (0, 15), $f = 5x + 11y = 165$
At (6, 12), $f = 5x + 11y = 162$
At (10.8, 0), $f = 5x + 11y = 54$

The maximum value is $f = 165$ at $x = 0, y = 15$.

Checkpoint Solutions

1. The values of f at the corners are found as follows.

At (0, 0), $f = 0$
At (2, 0), $f = 8$
At (3, 2), $f = 12 + 6 = 18$
At (0, 4), $f = 12$

The maximum value of f is 18 at $x = 3, y = 2$, and the minimum value is $f = 0$ at $x = 0, y = 0$.

2. The graph of the feasible region is shown in Figure 4.15. The values of g at the corners are found as follows:

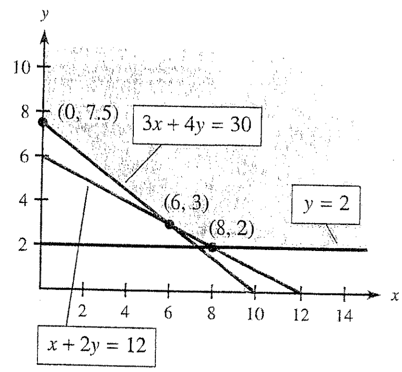


Figure 4.15

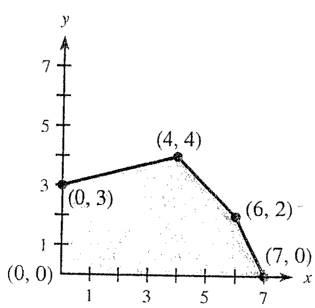
At (0, 7.5), $g = 30$
At (6, 3), $g = 18 + 12 = 30$
At (8, 2), $g = 24 + 8 = 32$

The minimum value of g is 30 at both (0, 7.5) and (6, 3). Thus any point on the border joining (0, 7.5) and (6, 3) will give the minimum value 30. For example, (2, 6) is on this border and gives the value $6 + 24 = 30$. The maximum value of g does not exist; g can be made arbitrarily large on this feasible region.

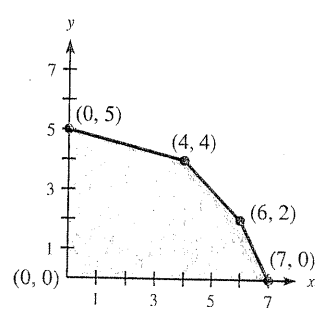
4.2 Exercises

In Problems 1–6, use the given feasible region determined by the constraint inequalities to find the maximum and minimum of the given objective function (if they exist).

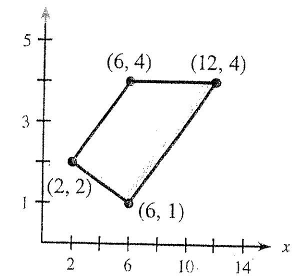
1. $C = 2x + 3y$



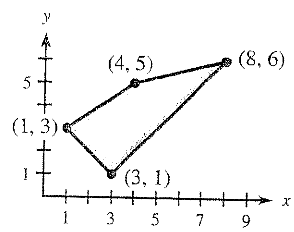
2. $f = 6x + 4y$



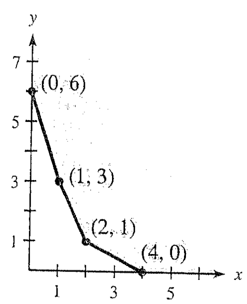
3. $C = 5x + 2y$



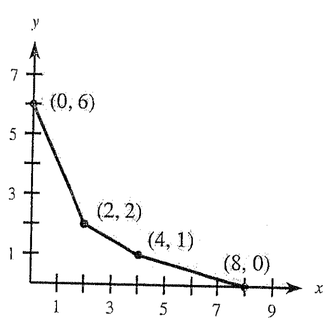
4. $C = 4x + 7y$



5. $f = 3x + 4y$

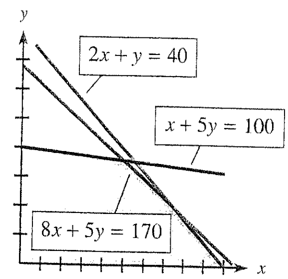


6. $f = 4x + 5y$

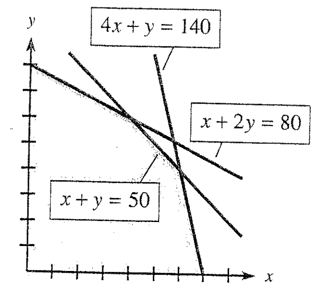


In each of Problems 7–10, the graph of the feasible region is shown. Find the corners of each feasible region, and then find the maximum and minimum of the given objective function (if they exist).

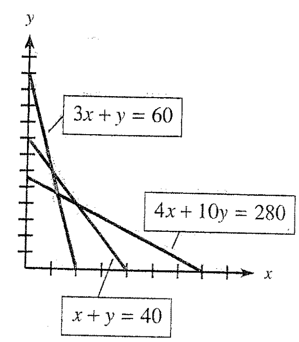
7. $f = 3x + 2y$



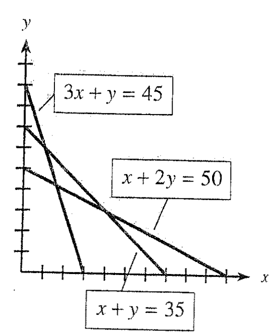
8. $f = 5x + 8y$



9. $g = 3x + 2y$



10. $g = x + 3y$



In Problems 11–14, find the indicated maximum or minimum value of the objective function in the linear programming problem. Note that the feasible regions for these problems are found in the answers to Problems 19, 20, 23, and 24 in the 4.1 Exercises.

11. Maximize $f = 3x + 2y$ subject to

$$\begin{aligned} x + 2y &\le 48 \\ x + y &\le 30 \\ 2x + y &\le 50 \\ x \ge 0, y &\ge 0 \end{aligned}$$

12. Maximize $f = 7x + 10y$ subject to

$$\begin{aligned} 3x + y &\le 9 \\ 3x + 2y &\le 12 \\ x + 2y &\le 8 \\ x \ge 0, y &\ge 0 \end{aligned}$$

13. Minimize $g = 12x + 48y$ subject to

$$\begin{aligned} x + 3y &\ge 3 \\ 2x + 3y &\ge 5 \\ 2x + y &\ge 3 \\ x \ge 0, y &\ge 0 \end{aligned}$$

14. Minimize $g = 12x + 8y$ subject to

$$\begin{aligned} x + 2y &\ge 10 \\ 2x + y &\ge 11 \\ x + y &\ge 9 \\ x \ge 0, y &\ge 0 \end{aligned}$$

In Problems 15–26, solve the following linear programming problems.

15. Maximize $f = 3x + 4y$ subject to

$$\begin{aligned} x + y &\le 6 \\ 2x + y &\le 10 \\ y &\le 4 \\ x \ge 0, y &\ge 0 \end{aligned}$$

16. Maximize $f = x + 3y$ subject to

$$\begin{aligned} x + 4y &\le 12 \\ y &\le 2 \\ x + y &\le 9 \\ x \ge 0, y &\ge 0 \end{aligned}$$

17. Maximize $f = 2x + 6y$ subject to

$$\begin{aligned} x + y &\le 7 \\ 2x + y &\le 12 \\ x + 3y &\le 15 \\ x \ge 0, y &\ge 0 \end{aligned}$$

18. Maximize $f = 4x + 2y$ subject to

$$\begin{aligned} x + 2y &\le 20 \\ x + y &\le 12 \\ 4x + y &\le 36 \\ x \ge 0, y &\ge 0 \end{aligned}$$

19. Minimize $g = 7x + 6y$ subject to

$$\begin{aligned} 5x + 2y &\ge 16 \\ 3x + 7y &\ge 27 \\ x \ge 0, y &\ge 0 \end{aligned}$$

20. Minimize $g = 22x + 17y$ subject to

$$\begin{aligned} 8x + 5y &\ge 100 \\ 12x + 25y &\ge 360 \\ x \ge 0, y &\ge 0 \end{aligned}$$

21. Minimize $g = 3x + y$ subject to

$$\begin{aligned} 4x + y &\ge 11 \\ 3x + 2y &\ge 12 \\ x \ge 0, y &\ge 0 \end{aligned}$$

22. Minimize $g = 50x + 70y$ subject to

$$\begin{aligned} 11x + 15y &\ge 225 \\ x + 3y &\ge 27 \\ x \ge 0, y &\ge 0 \end{aligned}$$

23. Maximize $f = x + 2y$ subject to

$$\begin{aligned} x + y &\ge 4 \\ 2x + y &\le 8 \\ y &\le 4 \end{aligned}$$

24. Maximize $f = 3x + 5y$ subject to

$$\begin{aligned} 2x + 4y &\ge 8 \\ 3x + y &\le 7 \\ x \ge 0, y &\le 4 \end{aligned}$$

25. Minimize $g = 40x + 25y$ subject to

$$\begin{aligned} x + y &\ge 100 \\ -x + y &\le 20 \\ -2x + 3y &\ge 30 \\ x \ge 0, y &\ge 0 \end{aligned}$$

26. Minimize $g = 3x + 8y$ subject to

$$\begin{aligned} 4x - 5y &\ge 50 \\ -x + 2y &\ge 4 \\ x + y &\le 80 \\ x \ge 0, y &\ge 0 \end{aligned}$$