4.1 Linear Inequalities in Two Variables

OBJECTIVES

- To graph linear inequalities in two variables
- To solve systems of linear inequalities in two variables

Application Preview

A farm co-op has 6000 acres available to plant with corn and soybeans. Each acre of corn requires 9 gallons of fertilizer/herbicide and 3/4 hour of labor to harvest. Each acre of soybeans requires 3 gallons of fertilizer/herbicide and 1 hour of labor to harvest. The co-op has available at most 40,500 gallons of fertilizer/herbicide and at most 5250 hours of labor for harvesting. The number of acres of each crop is limited (constrained) by the available resources: land, fertilizer/herbicide, and labor for harvesting. Finding the values that satisfy all these constraints at the same time is called solving the system of inequalities. (See Example 4.)

One Linear Inequality in Two Variables

Before we look at systems of inequalities, we will discuss solutions of one inequality in two variables, such as $y < x$. The solutions of this inequality are the ordered pairs (x, y) that satisfy the inequality. Thus $(1, 0)$, $(3, 2)$, $(0, -1)$, and $(-2, -5)$ are solutions of $y < x$, but $(3, 7)$, $(-4, -3)$ and $(2, 2)$ are not.

The graph of $y < x$ consists of all points in which the y-coordinate is less than the x-coordinate. The graph of the region $y < x$ can be found by graphing the line $y = x$ (as a dashed line, because the given inequality does not include $y = x$). This line separates the xy-plane into two **half-planes,** $y < x$ and $y > x$. We can determine which half-plane is the solution region by selecting as a **test point** any point **not on the line;** let's choose $(2, 0)$. Because the coordinates of this test point satisfy the inequality $y < x$, the half-plane containing this point is the solution region for $y < x$. (See Figure 4.1.) If the coordinates of the test point do not satisfy the inequality, then the other half-plane is the solution region. For example, say we had chosen $(0, 4)$ as our test point. Its coordinates do not satisfy $y < x$, so the half-plane that does *not* contain $(0, 4)$ is the solution region. (Note that we get the same region.)

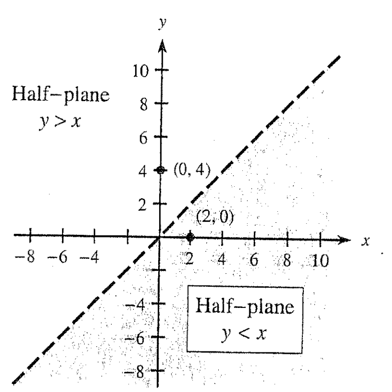


Figure 4.1

EXAMPLE 1 ***Graphing Inequalities***

Graph the inequality $4x - 2y \le 6$.

Solution

First we graph the line $4x - 2y = 6$, or (equivalently) $y = 2x - 3$, as a solid line, because points lying on the line satisfy the given inequality. Next we pick a test point that is not on the line. If we use $(0, 0)$, we see that its coordinates satisfy $4x - 2y \le 6$—that is,

$y \geq 2x - 3$. Hence the solution region is the line $y = 2x - 3$ and the half-plane that contains the test point (0, 0). See Figure 4.2.

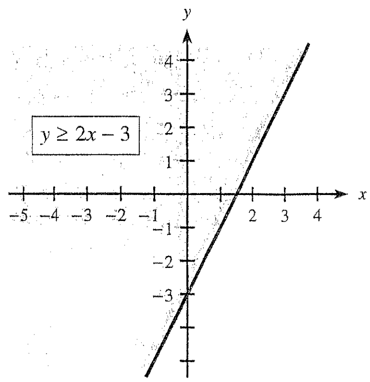


Figure 4.2

Technology Note

Graphing calculators can also be used to shade the solution region of an inequality. Figure 4.3 shows a graphing calculator window for the solution of $4x - 2y \leq 6$, which is equivalent to $4x - 6 \leq 2y$, or $y \geq 2x - 3$.

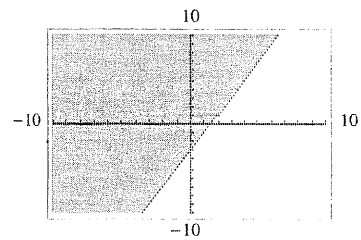


Figure 4.3

Systems of Linear Inequalities

If we have two inequalities in two variables, we can find the solutions that satisfy both inequalities. We call the inequalities a **system of inequalities,** and the solution of the system can be found by finding the intersection of the solution sets of the two inequalities.

The solution set of the system of inequalities can be found by graphing the inequalities on the same set of axes and noting their points of intersection.

● EXAMPLE 2 ***Graphical Solution of a System of Inequalities***

Graph the solution of the system

$$\begin{cases} 3x - 2y \geq 4 \\ x + y - 3 > 0 \end{cases}$$

Solution

Begin by graphing the equations $3x - 2y = 4$ and $x + y = 3$ (from $x + y - 3 = 0$) by the intercept method: find y when $x = 0$ and find x when $y = 0$.

$3x - 2y = 4$

x	y
0	−2
4/3	0

$x + y = 3$

x	y
0	3
3	0

We graph $3x - 2y = 4$ as a solid line and $x + y = 3$ as a dashed line (see Figure 4.4(a)). We use any point not on either line as a test point; let's use (0, 0). Note that the coordinates of (0, 0) do not satisfy either $3x - 2y \geq 4$ or $x + y - 3 > 0$. Thus the solution region for each individual inequality is the half-plane that does not contain the point (0, 0). Figure 4.4(b) indicates the half-plane solution for each inequality with arrows pointing from the line into the desired half-plane (away from the test point). The points that satisfy both of these inequalities lie in the intersection of the two individual solution regions, shown in Figure 4.4(c). This **solution region** is the graph of the solution of this system of inequalities.

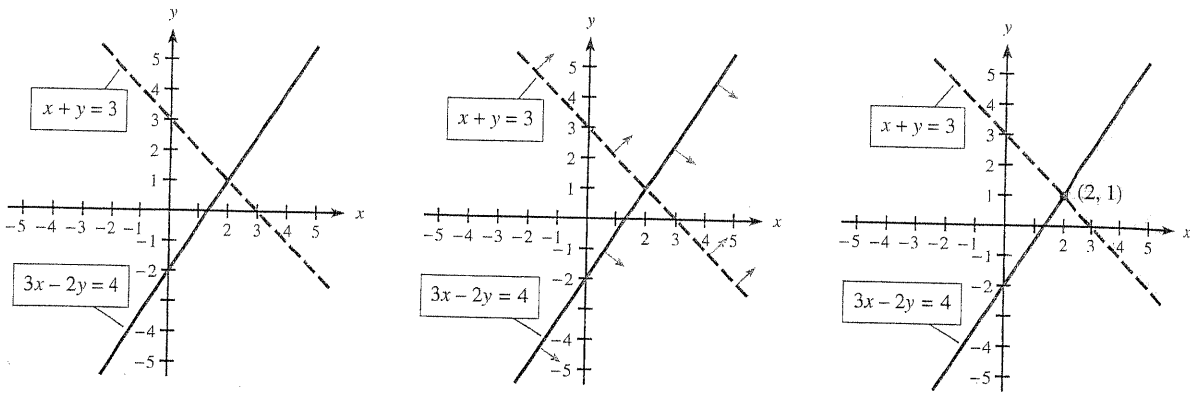


Figure 4.4 (a) (b) (c)

The point (2, 1) in Figure 4.4(c), where the two regions form a "corner," is found by solving the *equations* $3x - 2y = 4$ and $x + y = 3$ simultaneously.

EXAMPLE 3 *Graphical Solution of a System of Inequalities*

Graph the solution of the system

$$\begin{cases} x + 2y \leq 10 \\ 2x + \ \ y \leq 14 \\ x \geq 0, y \geq 0 \end{cases}$$

Solution

The two inequalities $x \geq 0$ and $y \geq 0$ restrict the solution to Quadrant I (and the axes bounding Quadrant I).

We seek points in the first quadrant (on or above $y = 0$ and on or to the right of $x = 0$) that satisfy $x + 2y \leq 10$ *and* $2x + y \leq 14$. We can write these inequalities in their equivalent forms $y \leq 5 - \frac{1}{2}x$ and $y \leq 14 - 2x$. The points that satisfy these inequalities (in the first quadrant) are shown by the shaded area in Figure 4.5. We can observe from the graph that the points (0, 0), (7, 0), and (0, 5) are corners of the solution region. The corner (6, 2) is found by solving the *equations* $y = 5 - \frac{1}{2}x$ and $y = 14 - 2x$ simultaneously as follows:

$$5 - \frac{1}{2}x = 14 - 2x$$

$$\frac{3}{2}x = 9$$

$$x = 6$$

$$y = 2$$

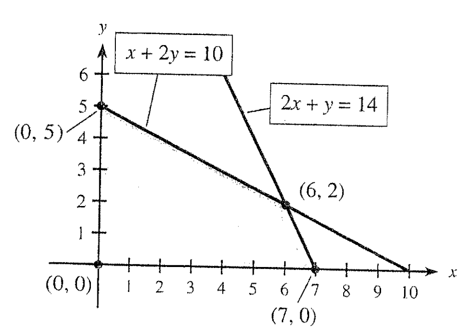


Figure 4.5

We will see that the corners of the solution region are important in solving linear programming problems.

Many applications restrict the variables to be nonnegative (such as $x \geq 0$ and $y \geq 0$ in Example 3). As we noted, the effect of this restriction is to limit the solution to Quadrant I and the axes bounding Quadrant I.

EXAMPLE 4 *Land Management (Application Preview)*

A farm co-op has 6000 acres available to plant with corn and soybeans. Each acre of corn requires 9 gallons of fertilizer/herbicide and 3/4 hour of labor to harvest. Each acre of soybeans requires 3 gallons of fertilizer/herbicide and 1 hour of labor to harvest. The co-op has available at most 40,500 gallons of fertilizer/herbicide and at most 5250 hours of labor for harvesting. The number of acres of each crop is limited (constrained) by the available resources: land, fertilizer/herbicide, and labor for harvesting. Write the system of inequalities that describes the constraints and graph the solution region for the system.

Solution

If x represents the number of acres of corn and y represents the number of acres of soybeans, then the total acres planted is limited by $x + y \leq 6000$. The limitation on fertilizer/herbicide is given by $9x + 3y \leq 40{,}500$, and the labor constraint is given by $\frac{3}{4}x + y \leq 5250$. Because the number of acres planted in each crop must be nonnegative, the system of inequalities that describes the constraints is as follows.

$$\begin{aligned} x + y &\leq 6{,}000 && (1)\\ 9x + 3y &\leq 40{,}500 && (2)\\ \frac{3}{4}x + y &\leq 5{,}250 && (3)\\ x \geq 0, y &\geq 0 \end{aligned}$$

The intercepts of the lines associated with the inequalities are:

(6000, 0) and (0, 6000) for the line $x + y = 6000$ (from inequality (1))

(4500, 0) and (0, 13,500) for the line $9x + 3y = 40{,}500$ (from inequality (2))

(7000, 0) and (0, 5250) for the line $\frac{3}{4}x + y = 5250$ (from inequality (3))

The solution region is shaded in Figure 4.6, with three of the corners at (0, 0), (4500, 0), and (0, 5250). The corner (3750, 2250) is found by solving simultaneously, as follows.

$$\begin{cases} 9x + 3y = 40{,}500 & (4)\\ x + y = 6{,}000 & (5) \end{cases}$$

$$\begin{aligned} &9x + 3y = 40{,}500 && (4)\\ &-(3x + 3y = 18{,}000) && [3 \times \text{Eq. } (5)]\\ \hline &6x = 22{,}500 \end{aligned}$$

$$x = \frac{22{,}500}{6} = 3750$$

$$y = 6000 - x = 2250$$

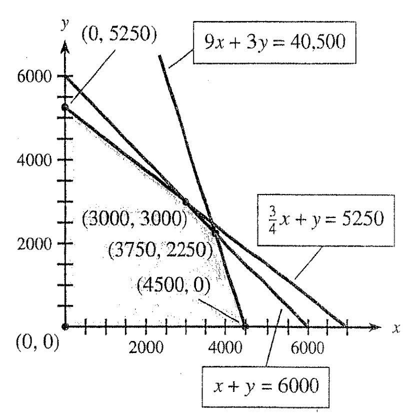


Figure 4.6

The corner (3000, 3000) is found similarly by solving simultaneously with $x + y = 6000$ and $\frac{3}{4}x + y = 5250$.

Any point in the shaded region of Figure 4.6 represents a possible number of acres of corn and of soybeans that the co-op could plant, treat with fertilizer/herbicide, and harvest.

Checkpoint

1. Graph the region determined by the inequalities

$$2x + 3y \leq 12$$
$$4x + 2y \leq 16$$
$$x \geq 0, y \geq 0$$

2. Determine the corners of the region.

EXAMPLE 5 *Manufacturing Constraints*

CDF Appliances has assembly plants in Atlanta and Fort Worth, where they produce a variety of kitchen appliances, including a 12-cup coffee maker and a cappuccino machine. At the Atlanta plant, 160 of the 12-cup models and 200 of the cappuccino machines can be assembled each hour. At the Fort Worth plant, 800 of the 12-cup models and 200 of the cappuccino machines can be assembled each hour. CDF Appliances expects orders for at least 64,000 of the 12-cup models and at least 40,000 of the cappuccino machines. At each plant, the number of assembly hours available for these two appliances is constrained by each plant's capacity and the need to fill the orders. Write the system of inequalities that describes these assembly plant constraints, and graph the solution region for the system.

Solution

Let x be the number of assembly hours at the Atlanta plant, and let y be the number of assembly hours at the Fort Worth plant. The production capabilities of each facility and the anticipated orders are summarized in the following table.

	Atlanta	Fort Worth	Needed
12-cup	160/hr.	800/hr.	At least 64,000
Cappuccino machine	200/hr.	200/hr.	At least 40,000

This table and the fact that both x and y must be nonnegative gives the following constraints.

$$160x + 800y \geq 64{,}000$$
$$200x + 200y \geq 40{,}000$$
$$x \geq 0, y \geq 0$$

As before, the solution is restricted to Quadrant I and the axes that bound Quadrant I. The equation $160x + 800y = 64{,}000$ can be graphed with x- and y-intercepts. The test point (0, 0) does not satisfy $160x + 800y \geq 64{,}000$, so the region lies above the line (see Figure 4.7(a)). In like manner, the region satisfying $200x + 200y \geq 40{,}000$ lies above the line $200x + 200y = 40{,}000$, and the solution region for this system is shown in Figure 4.7(b).

The corner (150, 50) is found by solving simultaneously as follows.

$$\begin{cases} 160x + 800y = 64{,}000 \\ 200x + 200y = 40{,}000 \end{cases} \quad \text{is equivalent to} \quad \begin{cases} x + 5y = 400 & (1) \\ x + y = 200 & (2) \end{cases}$$

Finding equation (1) minus equation (2) gives one equation in one variable and allows us to complete the solution.

$$4y = 200, \quad \text{so} \quad y = 200/4 = 50 \quad \text{and} \quad x = 200 - y = 150$$

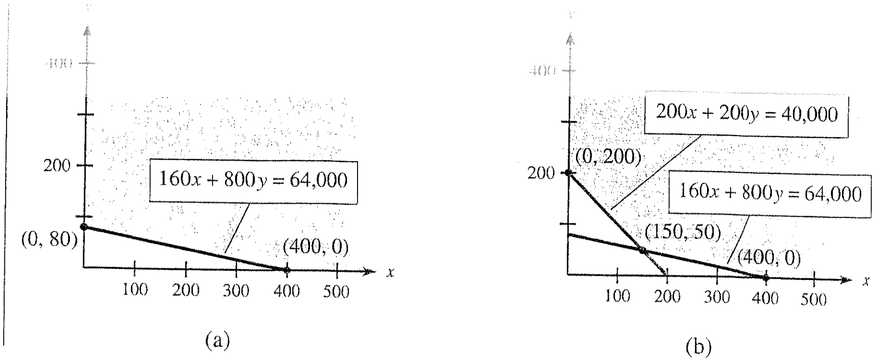


Figure 4.7

Many computer programs save considerable time and energy in determining regions that satisfy a system of inequalities. Graphing calculators can also be used in the graphical solution of a system of inequalities.

● EXAMPLE 6 *Solution Region with Technology*

Use a graphing utility to find the following.

(a) Find the region determined by the inequalities below.

$$5x + 2y \le 54$$
$$2x + 4y \le 60$$
$$x \ge 0, y \ge 0$$

(b) Find the corners of this region.

Solution

(a) We write the inequalities above as equations, solved for y.

$$5x + 2y = 54 \Rightarrow 2y = 54 - 5x$$
$$\Rightarrow y = 27 - \frac{5}{2}x$$
$$2x + 4y = 60 \Rightarrow 4y = 60 - 2x$$
$$\Rightarrow y = 15 - \frac{1}{2}x$$

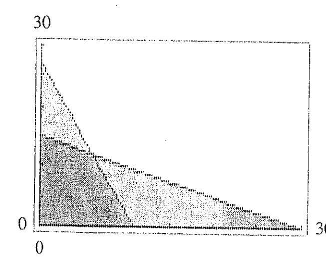
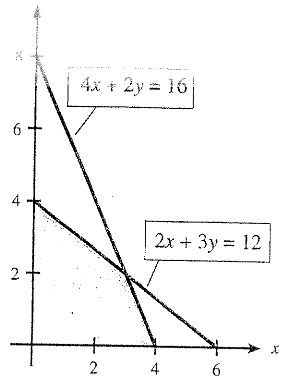


Figure 4.8

Graphing these equations with a graphing calculator and using shading shows the region satisfying the inequalities (see Figure 4.8).

(b) Some graphing utilities can determine the points of intersection of two equations with an INTERSECT command. This command can be used to find the intersection of the pair of equations from (a). With other graphing utilities, using the command SOLVER or INTERSECT with pairs of lines that form the borders of this region will give the points where the boundaries intersect. These points, (0, 0), (0, 15), (6, 12), and (10.8, 0), can also be found algebraically. These points are the corners of the solution region. These corners will be important to us in the graphical solutions of linear programming problems in the next section.

Checkpoint Solutions



2. The graph shows corners at (0, 0), (0, 4), and (4, 0). A corner also occurs where $2x + 3y = 12$ and $4x + 2y = 16$ intersect. The point of intersection is $x = 3$, $y = 2$, so (3, 2) is a corner.

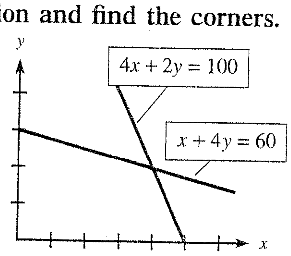
4.1 Exercises

In Problems 1–6, graph each inequality.

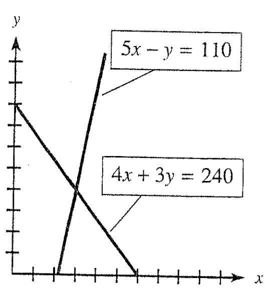
1. $y \le 2x - 1$
2. $y \ge 4x - 5$
3. $\frac{x}{2} + \frac{y}{4} < 1$
4. $x - \frac{y}{3} < \frac{-2}{3}$
5. $0.4x \ge 0.8$
6. $\frac{-y}{8} > \frac{1}{4}$

In Problems 7–12, the graph of the boundary equations for each system of inequalities is shown with that system. Locate the solution region and find the corners.

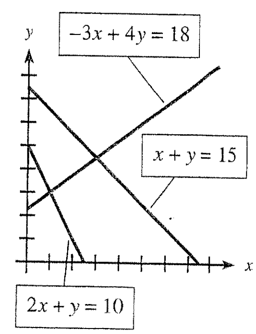
7. $\begin{cases} x + 4y \le 60 \\ 4x + 2y \le 100 \\ x \ge 0 \\ y \ge 0 \end{cases}$



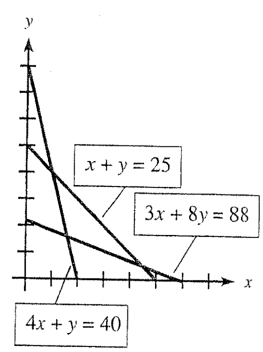
8. $\begin{cases} 4x + 3y \le 240 \\ 5x - y \le 110 \\ x \ge 0 \\ y \ge 0 \end{cases}$



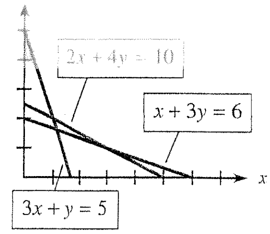
9. $\begin{cases} -3x + 4y \le 18 \\ 2x + y \ge 10 \\ x + y \le 15 \\ x \ge 0, y \ge 0 \end{cases}$



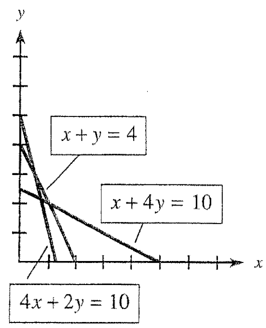
10. $\begin{cases} x + y \le 25 \\ 4x + y \le 40 \\ 3x + 8y \ge 88 \\ x \ge 0, y \ge 0 \end{cases}$



11. $\begin{cases} x + 3y \geq 6 \\ 2x + 4y \geq 10 \\ 3x + y \geq 5 \\ x \geq 0 \\ y \geq 0 \end{cases}$



12. $\begin{cases} x + 4y \geq 10 \\ 4x + 2y \geq 10 \\ x + y \geq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$



In Problems 13–26, graph the solution of each system of inequalities.

13. $\begin{cases} y < 2x \\ y > x - 1 \end{cases}$

14. $\begin{cases} y > 3x - 4 \\ y < 2x + 3 \end{cases}$

15. $\begin{cases} 2x + y < 3 \\ x - 2y \geq -1 \end{cases}$

16. $\begin{cases} 3x + y > 4 \\ x - 2y < -1 \end{cases}$

17. $\begin{cases} x + 5y \leq 200 \\ 2x + 3y \leq 134 \\ x \geq 0, y \geq 0 \end{cases}$

18. $\begin{cases} -x + y \leq 2 \\ x + 2y \leq 10 \\ 3x + y \leq 15 \\ x \geq 0, y \geq 0 \end{cases}$

19. $\begin{cases} x + 2y \leq 48 \\ x + y \leq 30 \\ 2x + y \leq 50 \\ x \geq 0, y \geq 0 \end{cases}$

20. $\begin{cases} 3x + y \leq 9 \\ 3x + 2y \leq 12 \\ x + 2y \leq 8 \\ x \geq 0, y \geq 0 \end{cases}$

21. $\begin{cases} x + 2y \geq 19 \\ 3x + 2y \geq 29 \\ x \geq 0, y \geq 0 \end{cases}$

22. $\begin{cases} 4x + y \geq 12 \\ x + y \geq 9 \\ x + 3y \geq 15 \\ x \geq 0, y \geq 0 \end{cases}$

23. $\begin{cases} x + 3y \geq 3 \\ 2x + 3y \geq 5 \\ 2x + y \geq 3 \\ x \geq 0, y \geq 0 \end{cases}$

24. $\begin{cases} x + 2y \geq 10 \\ 2x + y \geq 11 \\ x + y \geq 9 \\ x \geq 0, y \geq 0 \end{cases}$

25. $\begin{cases} x + 2y \geq 20 \\ -3x + 2y \leq 4 \\ x \geq 12 \\ x \geq 0, y \geq 0 \end{cases}$

26. $\begin{cases} 3x + 2y \geq 75 \\ -3x + 5y \geq 30 \\ y \leq 40 \\ x \geq 0, y \geq 0 \end{cases}$

APPLICATIONS

27. *Management* The Wellbuilt Company produces two types of wood chippers, economy and deluxe. The deluxe model requires 3 hours to assemble and 1/2 hour to paint, and the economy model requires 2 hours to assemble and 1 hour to paint. The maximum number of assembly hours available is 24 per day, and the maximum number of painting hours available is 8 per day.
   (a) Write the system of inequalities that describes the constraints on the number of each type of wood chipper produced. Begin by identifying what x and y represent.
   (b) Graph the solution of the system of inequalities and find the corners of the solution region.

28. *Learning environments* An experiment that involves learning in animals requires placing white mice and rabbits into separate, controlled environments, environment I and environment II. The maximum amount of time available in environment I is 500 minutes, and the maximum amount of time available in environment II is 600 minutes. The white mice must spend 10 minutes in environment I and 25 minutes in environment II, and the rabbits must spend 15 minutes in environment I and 15 minutes in environment II.
   (a) Write a system of inequalities that describes the constraints on the number of each type of animal used in the experiment. Begin by identifying what x and y represent.
   (b) Graph the solution of the system of inequalities and find the corners of the solution region.

29. *Manufacturing* A company manufactures two types of electric hedge trimmers, one of which is cordless. The cord-type trimmer requires 2 hours to make, and the cordless model requires 4 hours. The company has only 800 work hours to use in manufacturing each day, and the packaging department can package only 300 trimmers per day.
   (a) Write the inequalities that describe the constraints on the number of each type of hedge trimmer produced. Begin by identifying what x and y represent.
   (b) Graph the region determined by these constraint inequalities.

30. *Manufacturing* A firm manufactures bumper bolts and fender bolts for antique cars. One machine can produce 130 fender bolts per hour, and another machine can produce 120 bumper bolts per hour. The combined

number of fender bolts and bumper bolts the packaging department can handle is 230 per hour.

(a) Write the inequalities that describe the constraints on the number of each type of bolt produced. Begin by identifying what x and y represent.

(b) Graph the region determined by these constraint inequalities.

31. *Advertising* Apex Motors manufactures luxury cars and sport utility vehicles. The most likely customers are high-income men and women, and company managers want to initiate an advertising campaign targeting these groups. They plan to run 1-minute spots on business/investment programs, where they can reach 7 million women and 4 million men from their target groups. They also plan 1-minute spots during sporting events, where they can reach 2 million women and 12 million men from their target groups. Apex feels that the ads must reach at least 30 million women and at least 28 million men who are prospective customers.

(a) Write the inequalities that describe the constraints on the number of each type of 1-minute spots needed to reach these target groups.

(b) Graph the region determined by these constraint inequalities.

32. *Manufacturing* The Video Star Company makes two different models of DVD players, which are assembled on two different assembly lines. Line 1 can assemble 30 units of the Star model and 40 units of the Prostar model per hour, and Line 2 can assemble 150 units of the Star model and 40 units of the Prostar model per hour. The company needs to produce at least 270 units of the Star model and 200 units of the Prostar model to fill an order.

(a) Write the inequalities that describe the production constraints on the number of each type of DVD player needed to fill the order.

(b) Graph the region determined by these constraint inequalities.

33. *Politics* A candidate wishes to use a combination of radio and television advertisements in her campaign. Research has shown that each 1-minute spot on television reaches 0.09 million people and each 1-minute spot on radio reaches 0.006 million. The candidate feels she must reach at least 2.16 million people, and she must buy a total of at least 80 minutes of advertisements.

(a) Write the inequalities that relate the number of each type of advertising to her needs.

(b) Graph the region determined by these constraint inequalities.

34. *Nutrition* In a hospital ward, the patients can be grouped into two general categories depending on their condition and the amount of solid foods they require in their diet. A combination of two diets is used for solid foods because they supply essential nutrients for recovery. The table below summarizes the patient groups and their minimum daily requirements.

	Diet A	Diet B	Daily Requirement
Group 1	4 oz per serving	1 oz per serving	26 oz
Group 2	2 oz per serving	1 oz per serving	18 oz

(a) Write the inequalities that describe how many servings of each diet are needed to provide the nutritional requirements.

(b) Graph the region determined by these constraint inequalities.

35. *Manufacturing* A sausage company makes two different kinds of hot dogs, regular and all beef. Each pound of all-beef hot dogs requires 0.75 lb of beef and 0.2 lb of spices, and each pound of regular hot dogs requires 0.18 lb of beef, 0.3 lb of pork, and 0.2 lb of spices. Suppliers can deliver at most 1020 lb of beef, at most 600 lb of pork, and at least 500 lb of spices.

(a) Write the inequalities that describe how many pounds of each type of hot dog can be produced.

(b) Graph the region determined by these constraint inequalities.

36. *Manufacturing* A cereal manufacturer makes two different kinds of cereal, Senior Citizen's Feast and Kids Go. Each pound of Senior Citizen's Feast requires 0.6 lb of wheat and 0.2 lb of vitamin-enriched syrup, and each pound of Kids Go requires 0.4 lb of wheat, 0.2 lb of sugar, and 0.2 lb of vitamin-enriched syrup. Suppliers can deliver at most 2800 lb of wheat, at most 800 lb of sugar, and at least 1000 lb of the vitamin-enriched syrup.

(a) Write the inequalities that describe how many pounds of each type of cereal can be made.

(b) Graph the region determined by these constraint inequalities.