46. When a stone is thrown upward, it follows a parabolic path given by a form of the equation

$$y = ax^2 + bx + c$$

If $y = 0$ represents ground level, find the equation of a stone that is thrown from ground level at $x = 0$ and lands on the ground 40 units away if the stone reaches a maximum height of 40 units. (*Hint:* Find the coordinates of the vertex of the parabola and two other points.)

47. *Health care* Many politicians are discussing national health insurance because health care costs are increasing so rapidly. Health care costs in the United States are given in the following table.

Year	Costs ($ millions)	Year	Costs ($ millions)
1960	26.7	1997	1093.9
1970	73.1	1998	1146.1
1980	245.8	1999	1210.7
1990	695.6	2000	1310.0
1995	987.0	2001	1425.5

Source: U.S. Department of Health and Human Services, Health Care Financing Administration, Office of National Health Statistics

(a) Plot the points from the table. Use x to represent the number of years since 1960 and y to represent costs in millions of dollars.
(b) What type of function appears to be the best fit for these points?
(c) Graph different quadratic functions of the form $y = ax^2 + c$ until you find a curve that is a reasonable fit for the points.
(d) Use your "model" function to predict health care costs in 2010.

48. *Social Security* In 1995, America's 45 million Social Security recipients received a 2.6% cost-of-living increase, the second smallest increase in nearly 20 years, a reflection of lower inflation. The percent increase can be described by the function

$$p(t) = -0.3375t^2 + 7.3t - 34.3625$$

where t is the number of years past 1980. (*Source:* Social Security Administration)
(a) Graph the function $y = p(t)$.
(b) From the graph, identify t-values for which $p(t) < 0$ and hence the model is not valid.
(c) From the graph, identify the maximum point on the graph of $p(t)$.
(d) In what year is the cost of living percent increase highest?

Union participation **The percentage of U.S. workers who belonged to unions for selected years from 1930 to 2000 can be described by $u(x) = -0.013x^2 + 1.62x - 20.33$, where x is the number of years past 1900 (*Source:* World Almanac, 2002). Use this function in Problems 49–51.**

49. Graph the function $u(x)$.
50. For what year does the function $u(x)$ indicate a maximum percentage of workers belonged to unions?
51. (a) For what years does the function $u(x)$ predict that 0% of U.S. workers will belong to unions?
(b) When can you guarantee that $u(x)$ can no longer be used to describe the percentage of U.S. workers who belong to unions?

2.3 Business Applications of Quadratic Functions

OBJECTIVES

- To graph quadratic supply and demand functions
- To find market equilibrium by using quadratic supply and demand functions
- To find break-even points by using quadratic cost and revenue functions
- To maximize quadratic revenue and profit functions

Application Preview

Suppose that the supply function for a product is given by the quadratic function $p = q^2 + 100$ and the demand function is given by $p = -20q + 2500$. Finding the market equilibrium involves solving a quadratic equation (see Example 1).

In this section, we graph quadratic supply and demand functions and find market equilibrium by solving supply and demand functions simultaneously using quadratic methods. We will also discuss quadratic revenue, cost, and profit functions, including break-even points and profit maximization.

Supply, Demand, and Market Equilibrium

The first-quadrant parts of parabolas or other quadratic equations are frequently used to represent supply and demand functions. For example, the first-quadrant part of $p = q^2 + q + 2$ (Figure 2.11(a) on the next page) may represent a supply curve, whereas the first-quadrant part of $q^2 + 2q + 6p - 23 = 0$ (Figure 2.11(b)) may represent a demand curve.

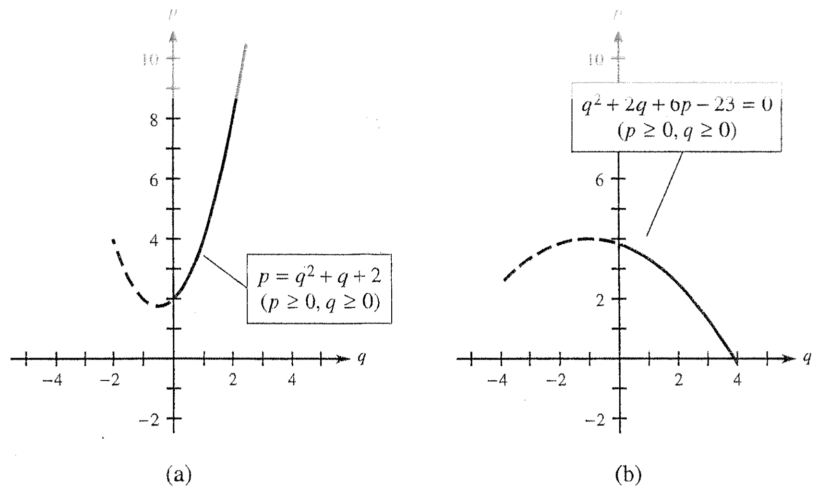

Figure 2.11

When quadratic equations are used to represent supply or demand curves, we can solve their equations simultaneously to find the market equilibrium as we did with linear supply and demand functions. As in Section 1.5, we can solve two equations in two variables by eliminating one variable and obtaining an equation in the other variable. When the functions are quadratic, the substitution method of solution is perhaps the best, and the resulting equation in one unknown will usually be quadratic.

EXAMPLE 1 *Supply and Demand (Application Preview)*

If the supply function for a commodity is given by $p = q^2 + 100$ and the demand function is given by $p = -20q + 2500$, find the point of market equilibrium.

Solution

At market equilibrium, both equations will have the same p-value. Thus substituting $q^2 + 100$ for p in $p = -20q + 2500$ yields

$$q^2 + 100 = -20q + 2500$$
$$q^2 + 20q - 2400 = 0$$
$$(q - 40)(q + 60) = 0$$
$$q = 40 \quad \text{or} \quad q = -60$$

Because a negative quantity has no meaning, the equilibrium point occurs when 40 units are sold, at (40, 1700). The graphs of the functions are shown (in the first quadrant only) in Figure 2.12.

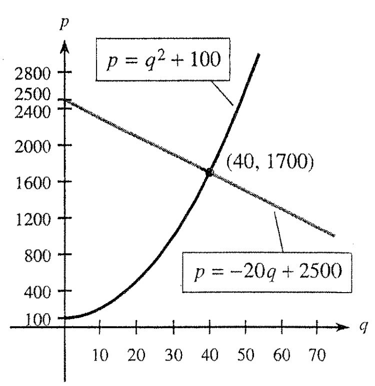

Figure 2.12

EXAMPLE 2 *Market Equilibrium*

If the demand function for a commodity is given by $p(q + 4) = 400$ and the supply function is given by $2p - q - 38 = 0$, find the market equilibrium.

Solution

Solving the supply equation $2p - q - 38 = 0$ for p gives $p = \frac{1}{2}q + 19$. Substituting for p in $p(q + 4) = 400$ gives

$$\left(\frac{1}{2}q + 19\right)(q + 4) = 400$$
$$\frac{1}{2}q^2 + 21q + 76 = 400$$
$$\frac{1}{2}q^2 + 21q - 324 = 0$$

Multiplying both sides of the equation by 2 yields $q^2 + 42q - 648 = 0$. Factoring gives

$$(q - 12)(q + 54) = 0$$
$$q = 12 \quad \text{or} \quad q = -54$$

Thus the market equilibrium occurs when 12 items are sold, at a price of $p = \frac{1}{2}(12) + 19 = \$25$ each. The graphs of the demand and supply functions are shown in Figure 2.13(a).

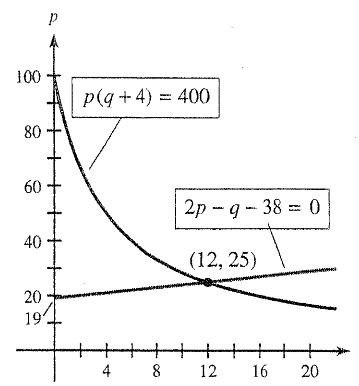

Figure 2.13 (a)

Graphing Utilities

Graphing utilities also can be used to sketch these graphs. A command such as INTERSECT could be used to determine points of intersection that give market equilibrium.

Figure 2.13(b) shows the graph of the supply and demand functions for the commodity in Example 2. Using the INTERSECT command gives the same market equilibrium point determined in Example 2 (see Figure 2.13(c)).

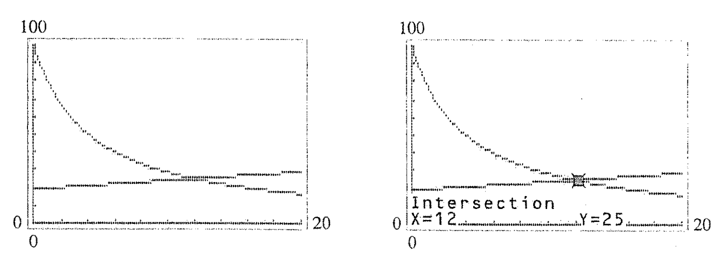

Figure 2.13 (*continued*) (b) (c)

Checkpoint

1. The point of intersection of the supply and demand functions is called ________.
2. If the demand and supply functions for a product are

$$p + \frac{1}{10}q^2 = 1000 \quad \text{and} \quad p = \frac{1}{10}q + 10$$

respectively, finding the market equilibrium point requires solution of what equation? Find the market equilibrium.

Break-Even Points and Maximization

In Chapter 1, "Linear Equations and Functions," we discussed linear total cost and total revenue functions. Many total revenue functions may be linear, but costs tend to increase sharply after a certain level of production. Thus functions other than linear functions, including quadratic functions, are used to predict the total costs of products.

For example, the monthly total cost curve for a commodity may be the parabola with equation $C(x) = 360 + 40x + 0.1x^2$. If the total revenue function is $R(x) = 60x$, we can find the break-even point by finding the quantity x that makes $C(x) = R(x)$. (See Figure 2.14.)

Setting $C(x) = R(x)$, we have

$$\begin{aligned} 360 + 40x + 0.1x^2 &= 60x \\ 0.1x^2 - 20x + 360 &= 0 \\ x^2 - 200x + 3600 &= 0 \\ (x - 20)(x - 180) &= 0 \\ x = 20 \qquad \text{or} \qquad x &= 180 \end{aligned}$$

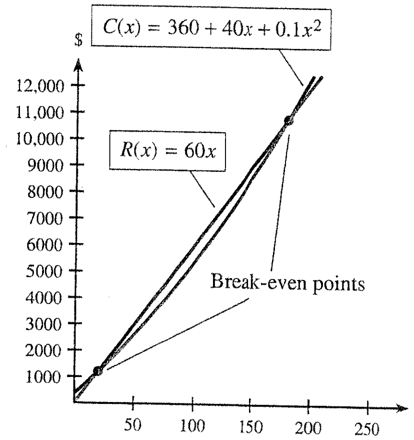

Figure 2.14

Thus $C(x) = R(x)$ at $x = 20$ and at $x = 180$. If 20 items are produced and sold, $C(x)$ and $R(x)$ are both \$1200; if 180 items are sold, $C(x)$ and $R(x)$ are both \$10,800. Thus there are two break-even points.

In a monopoly market, the revenue of a company is restricted by the demand for the product. In this case, the relationship between the price p of the product and the number of units sold x is described by the demand function $p = f(x)$, and the total revenue function for the product is given by

$$R = px = [f(x)]x$$

If, for example, the demand for a product is given by

$$p = 300 - x$$

where x is the number of units sold and p is the price, then the revenue function for this product is the quadratic function

$$R = px = (300 - x)x = 300x - x^2$$

EXAMPLE 3 ***Break-Even Point***

Suppose that in a monopoly market the total cost per week of producing a high-tech product is given by $C = 3600 + 100x + 2x^2$. Suppose further that the weekly demand function for this product is $p = 500 - 2x$. Find the number of units that will give the break-even point for the product.

Solution

The total cost function is $C(x) = 3600 + 100x + 2x^2$, and the total revenue function is $R(x) = (500 - 2x)x = 500x - 2x^2$.

Setting $C(x) = R(x)$ and solving for x gives

$$3600 + 100x + 2x^2 = 500x - 2x^2$$
$$4x^2 - 400x + 3600 = 0$$
$$x^2 - 100x + 900 = 0$$
$$(x - 90)(x - 10) = 0$$
$$x = 90 \quad \text{or} \quad x = 10$$

Does this mean the firm will break even at 10 units and at 90 units? Yes. Figure 2.15 shows the graphs of $C(x)$ and $R(x)$. From the graph we can observe that the firm makes a profit after $x = 10$ *until* $x = 90$, because $R(x) > C(x)$ in that interval. At $x = 90$, the profit is 0, and the firm loses money if it produces more than 90 units per week.

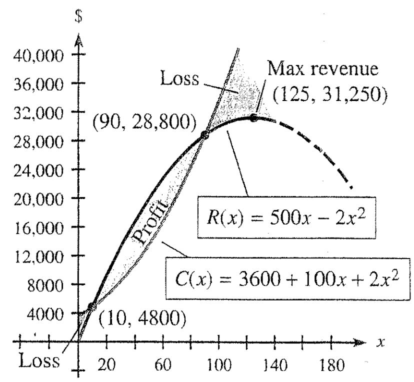

Figure 2.15

Checkpoint

3. The point of intersection of the revenue function and the cost function is called ________.
4. If $C(x) = 120x + 15{,}000$ and $R(x) = 370x - x^2$, finding the break-even points requires solution of what equation? Find the break-even points.

Note that for Example 3, the revenue function

$$R(x) = (500 - 2x)x = 500x - 2x^2$$

is a parabola that opens downward. Thus the vertex is the point at which revenue is maximum. We can locate this vertex by using the methods discussed in the previous section.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-500}{2(-2)} = \frac{500}{4} = 125 \text{ (units)}$$

It is interesting to note that when $x = 125$, the firm achieves its maximum revenue of

$$R(125) = 500(125) - 2(125)^2 = 31{,}250 \text{ (dollars)}$$

but the costs when $x = 125$ are

$$C(125) = 3600 + 100(125) + 2(125)^2 = 47{,}350 \text{ (dollars)}$$

which results in a loss. This illustrates that maximizing revenue is not a good goal. We should seek to maximize profit. Figure 2.15 shows that maximum profit will occur where the distance between the revenue and cost curves (i.e., $R(x) - C(x)$) is largest. This appears to be near $x = 50$, which is midway between the x-values of the break-even points. This is verified in the next example.

EXAMPLE 4 *Profit Maximization*

For the total cost function $C(x) = 3600 + 100x + 2x^2$ and the total revenue function $R(x) = 500x - 2x^2$ (from Example 3), find the number of units that maximizes profit and find the maximum profit.

Solution

Using Profit = Revenue − Cost, we can determine the profit function:

$$P(x) = (500x - 2x^2) - (3600 + 100x + 2x^2) = -3600 + 400x - 4x^2$$

This profit function is a parabola that opens downward, so the vertex will be the maximum point.

$$\text{Vertex: } x = \frac{-b}{2a} = \frac{-400}{2(-4)} = \frac{-400}{-8} = 50$$

Furthermore, when $x = 50$, we have

$$P(50) = -3600 + 400(50) - 4(50)^2 = 6400 \text{ (dollars)}$$

Thus, when 50 items are produced and sold, a maximum profit of \$6400 is made (see Figure 2.16).

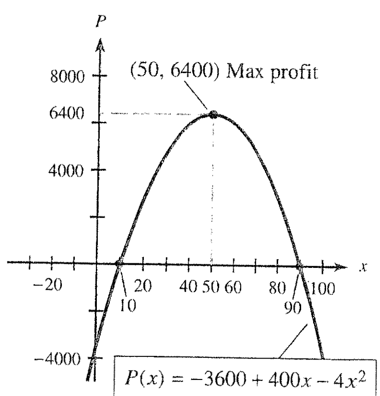

Figure 2.16

Figure 2.16 shows the break-even points at $x = 10$ and $x = 90$, and that the maximum profit occurs at the x-value midway between these x-values. This is reasonable because the graph of the profit function is a parabola, and the x-value of any parabola's vertex occurs midway between its x-intercepts.

It is important to note that the procedures for finding maximum revenue and profit in these examples depend on the fact that these functions are parabolas. Using methods discussed in Sections 2.1 and 2.2, we can use graphing utilities to locate maximum points, minimum points, and break-even points. For more general functions, procedures for finding maximum or minimum values are discussed in Chapter 10, "Applications of Derivatives."

Checkpoint Solutions

1. The market equilibrium point or market equilibrium
2. Demand: $p = -\frac{1}{10}q^2 + 1000$; supply: $p = \frac{1}{10}q + 10$

   Solution of $-\frac{1}{10}q^2 + 1000 = \frac{1}{10}q + 10$

$$-q^2 + 10{,}000 = q + 100$$
$$0 = q^2 + q - 9900$$
$$0 = (q + 100)(q - 99)$$
$$q = -100 \quad \text{or} \quad q = 99$$

   Thus market equilibrium occurs when $q = 99$ and $p = 9.9 + 10 = 19.90$.
3. The break-even point
4. Solution of $C(x) = R(x)$. That is, solution of $120x + 15{,}000 = 370x - x^2$, or $x^2 - 250x + 15{,}000 = 0$.

$$(x - 100)(x - 150) = 0$$
$$x = 100 \quad \text{or} \quad x = 150$$

   Thus the break-even points are when 100 units and 150 units are produced.

2.3 Exercises

SUPPLY, DEMAND, AND MARKET EQUILIBRIUM

1. Sketch the first-quadrant portions of the following on the same set of axes.
   (a) The supply function whose equation is $p = \frac{1}{4}q^2 + 10$
   (b) The demand function whose equation is $p = 86 - 6q - 3q^2$
   (c) Label the market equilibrium point.
   (d) Algebraically determine the equilibrium point for the supply and demand functions.
2. Sketch the first-quadrant portions of the following on the same set of axes.
   (a) The supply function whose equation is $p = q^2 + 8q + 16$
   (b) The demand function whose equation is $p = 216 - 2q$
   (c) Label the market equilibrium point.
   (d) Algebraically determine the equilibrium point for the supply and demand functions.
3. Sketch the first-quadrant portions of the following on the same set of axes.
   (a) The supply function whose equation is $p = 0.2q^2 + 0.4q + 1.8$
   (b) The demand function whose equation is $p = 9 - 0.2q - 0.1q^2$
   (c) Label the market equilibrium point.
   (d) Algebraically determine the market equilibrium point.
4. Sketch the first-quadrant portions of the following on the same set of axes.
   (a) The supply function whose equation is $p = q^2 + 8q + 22$
   (b) The demand function whose equation is $p = 198 - 4q - \frac{1}{4}q^2$
   (c) Label the market equilibrium point.
   (d) Algebraically determine the market equilibrium point for the supply and demand functions.
5. If the supply function for a commodity is $p = q^2 + 8q + 16$ and the demand function is $p = -3q^2 + 6q + 436$, find the equilibrium quantity and equilibrium price.
6. If the supply function for a commodity is $p = q^2 + 8q + 20$ and the demand function is $p = 100 - 4q - q^2$, find the equilibrium quantity and equilibrium price.
7. If the demand function for a commodity is given by the equation $p^2 + 4q = 1600$ and the supply function is given by the equation $300 - p^2 + 2q = 0$, find the equilibrium quantity and equilibrium price.
8. If the supply and demand functions for a commodity are given by $4p - q = 42$ and $(p + 2)q = 2100$, respectively, find the price that will result in market equilibrium.
9. If the supply and demand functions for a commodity are given by $p - q = 10$ and $q(2p - 10) = 2100$, what is the equilibrium price and what is the corresponding number of units supplied and demanded?

10. If the supply and demand functions for a certain product are given by the equations $2p - q + 6 = 0$ and $(p + q)(q + 10) = 3696$, respectively, find the price and quantity that give market equilibrium.
11. The supply function for a product is $2p - q - 10 = 0$, while the demand function for the same product is $(p + 10)(q + 30) = 7200$. Find the market equilibrium point.
12. The supply and demand for a product are given by $2p - q = 50$ and $pq = 100 + 20q$, respectively. Find the market equilibrium point.
13. For the product in Problem 11, if a \$22 tax is placed on production of the item, then the supplier passes this tax on by adding \$22 to his selling price. Find the new equilibrium point for this product when the tax is passed on. (The new supply function is given by $p = \frac{1}{2}q + 27$.)
14. For the product in Problem 12, if a \$12.50 tax is placed on production and passed through by the supplier, find the new equilibrium point.

BREAK-EVEN POINTS AND MAXIMIZATION

15. The total costs for a company are given by

$$C(x) = 2000 + 40x + x^2$$

and the total revenues are given by

$$R(x) = 130x$$

Find the break-even points.

16. If a firm has the following cost and revenue functions, find the break-even points.

$$C(x) = 3600 + 25x + \frac{1}{2}x^2,$$

$$R(x) = \left(175 - \frac{1}{2}x\right)x$$

17. If a company has total costs $C(x) = 15{,}000 + 35x + 0.1x^2$ and total revenues given by $R(x) = 385x - 0.9x^2$, find the break-even points.
18. If total costs are $C(x) = 1600 + 1500x$ and total revenues are $R(x) = 1600x - x^2$, find the break-even points.
19. Given that $P(x) = 11.5x - 0.1x^2 - 150$ and that production is restricted to fewer than 75 units, find the break-even points.
20. If the profit function for a firm is given by $P(x) = -1100 + 120x - x^2$ and limitations on space require that production is less than 100 units, find the break-even points.
21. Find the maximum revenue for the revenue function $R(x) = 385x - 0.9x^2$.
22. Find the maximum revenue for the revenue function $R(x) = 1600x - x^2$.
23. If, in a monopoly market, the demand for a product is $p = 175 - 0.50x$ and the revenue function is $R = px$, where x is the number of units sold, what price will maximize revenue?
24. If, in a monopoly market, the demand for a product is $p = 1600 - x$ and the revenue is $R = px$, where x is the number of units sold, what price will maximize revenue?
25. The profit function for a certain commodity is $P(x) = 110x - x^2 - 1000$. Find the level of production that yields maximum profit, and find the maximum profit.
26. The profit function for a firm making widgets is $P(x) = 88x - x^2 - 1200$. Find the number of units at which maximum profit is achieved, and find the maximum profit.
27. (a) Graph the profit function $P(x) = 80x - 0.1x^2 - 7000$.
    (b) Find the vertex of the graph. Is it a maximum point or a minimum point?
    (c) Is the average rate of change of this function from $x = a < 400$ to $x = 400$ positive or negative?
    (d) Is the average rate of change of this function from $x = 400$ to $x = a > 400$ positive or negative?
    (e) Does the average rate of change of the profit get closer to or farther from 0 when a is closer to 400?
28. (a) Graph the profit function $P(x) = 50x - 0.2x^2 - 2000$.
    (b) Find the vertex of the graph. Is it a maximum point or a minimum point?
    (c) Is the average rate of change of this function from $x = a < 125$ to $x = 125$ positive or negative?
    (d) Is the average rate of change of this function from $x = 125$ to $x = a > 125$ positive or negative?
    (e) Does the average rate of change of the profit get closer to or farther from 0 when a is closer to 125?
29. (a) Form the profit function for the cost and revenue functions in Problem 17, and find the maximum profit.
    (b) Compare the level of production to maximize profit with the level to maximize revenue (see Problem 21). Do they agree?
    (c) How do the break-even points compare with the zeros of $P(x)$?
30. (a) Form the profit function for the cost and revenue functions in Problem 18, and find the maximum profit.

(b) Compare the level of production to maximize profit with the level to maximize revenue (see Problem 22). Do they agree?
(c) How do the break-even points compare with the zeros of $P(x)$?

31. Suppose a company has fixed costs of $28,000 and variable costs of $\frac{2}{5}x + 222$ dollars per unit, where x is the total number of units produced. Suppose further that the selling price of its product is $1250 - \frac{3}{5}x$ dollars per unit.
(a) Find the break-even points.
(b) Find the maximum revenue.
(c) Form the profit function from the cost and revenue functions and find maximum profit.
(d) What price will maximize the profit?

32. Suppose a company has fixed costs of $300 and variable costs of $\frac{3}{4}x + 1460$ dollars per unit, where x is the total number of units produced. Suppose further that the selling price of its product is $1500 - \frac{1}{4}x$ dollars per unit.
(a) Find the break-even points.
(b) Find the maximum revenue.
(c) Form the profit function from the cost and revenue functions and find maximum profit.
(d) What price will maximize the profit?

33. The following table gives the total revenues of Cablenet Communications for selected years.

Year	Total Revenues (millions)
1997	$63.13
1998	$69.906
1999	$60.53
2001	$61.1
2002	$62.191
2003	$63.089
2004	$64.904
2005	$67.156

Suppose the data can be described by the equation

$$R(t) = 0.253t^2 - 4.03t + 76.84$$

where t is the number of years past 1992.
(a) Use the function to find the year in which revenue was minimum and find the minimum predicted revenue.
(b) Check the result from (a) against the data in the table.
(c) Graph $R(t)$ and the data points from the table.
(d) Write a sentence to describe how well the function fits the data.

The data in the table give sales revenues and costs and expenses for Continental Divide Mining for various years. Use this table in Problems 34 and 35.

Year	Sales Revenue (millions)	Costs and Expenses (millions)
1995	$2.6155	$2.4105
1996	2.7474	2.4412
1997	2.934	2.6378
1998	3.3131	2.9447
1999	3.9769	3.5344
2000	4.5494	3.8171
2001	4.8949	4.2587
2002	5.1686	4.8769
2003	4.9593	4.9088
2004	5.0913	4.6771
2005	4.7489	4.9025

34. Assume that sales revenues for Continental Divide Mining can be described by

$$R(t) = -0.031t^2 + 0.776t + 0.179$$

where t is the number of years past 1992.
(a) Use the function to determine the year in which maximum revenue occurs and the maximum revenue it predicts.
(b) Check the result from (a) against the data in the table.
(c) Graph $R(t)$ and the data points from the table.
(d) Write a sentence to describe how well the function fits the data.

35. Assume that costs and expenses for Continental Divide Mining can be described by

$$C(t) = -0.012t^2 + 0.492t + 0.725$$

where t is the number of years past 1992.
(a) Use $R(t)$ as given in Problem 34 and form the profit function (as a function of time).
(b) Use the function from (a) to find the year in which maximum profit occurs.
(c) Graph the profit function from (b) and the data points from the table.
(d) Through the decade from 2000 to 2010, does the function project increasing or decreasing profits? Do the data support this trend (as far as it goes)?
(e) How might management respond to this kind of projection?