counseling and $200 for emergencies, and type C clients require an average of $300 for counseling and $100 for emergencies. How many of each type of client can be served?

54. If funding for counseling is cut to $135,000 and funding for emergency food and shelter is cut to $90,000, only 450 clients can be served. How many of each type can be served in this case? (See Problem 53.)

1.6 Applications of Functions in Business and Economics

OBJECTIVES

- To formulate and evaluate total cost, total revenue, and profit functions
- To find marginal cost, revenue, and profit, given linear total cost, total revenue, and profit functions
- To write the equations of linear total cost, total revenue, and profit functions by using information given about the functions
- To find break-even points
- To evaluate and graph supply and demand functions
- To find market equilibrium

Application Preview

Suppose a firm manufactures MP3 players and sells them for $50 each, with costs incurred in the production and sale equal to $200,000 plus $10 for each unit produced and sold. Forming the total cost, total revenue, and profit as functions of the quantity x that is produced and sold (see Example 1) is called the **theory of the firm.** We will also discuss **market analysis,** in which supply and demand are found as functions of price, and market equilibrium is found.

Total Cost, Total Revenue, and Profit

The **profit** a firm makes on its product is the difference between the amount it receives from sales (its revenue) and its cost. If x units are produced and sold, we can write

$$P(x) = R(x) - C(x)$$

where

$P(x) =$ profit from sale of x units
$R(x) =$ total revenue from sale of x units
$C(x) =$ total cost of production and sale of x units*

In general, **revenue** is found by using the equation

$$\text{Revenue} = (\text{price per unit})(\text{number of units})$$

The **cost** is composed of two parts, fixed costs and variable costs. **Fixed costs** *(FC)*, such as depreciation, rent, utilities, and so on, remain constant regardless of the number of units produced. **Variable costs** *(VC)* are those directly related to the number of units produced. Thus the cost is found by using the equation

$$\text{Cost} = \text{variable costs} + \text{fixed costs}$$

EXAMPLE 1 *Cost, Revenue, and Profit (Application Preview)*

Suppose that a firm manufactures MP3 players and sells them for $50 each. The costs incurred in the production and sale of the MP3 players are $200,000 plus $10 for each player produced and sold. Write the profit function for the production and sale of x players.

Solution

The total revenue for x MP3 players is $50x$, so the total revenue function is $R(x) = 50x$. The fixed costs are $200,000, so the total cost for x players is $10x + 200{,}000$. Hence, $C(x) = 10x + 200{,}000$. The profit function is given by $P(x) = R(x) - C(x)$. Thus,

$$P(x) = 50x - (10x + 200{,}000)$$
$$P(x) = 40x - 200{,}000$$

Figure 1.41 shows the graphs of $R(x)$, $C(x)$, and $P(x)$.

*The symbols generally used in economics for total cost, total revenue, and profit are *TC*, *TR*, and π, respectively. In order to avoid confusion, especially with the use of π as a variable, we do not use these symbols.

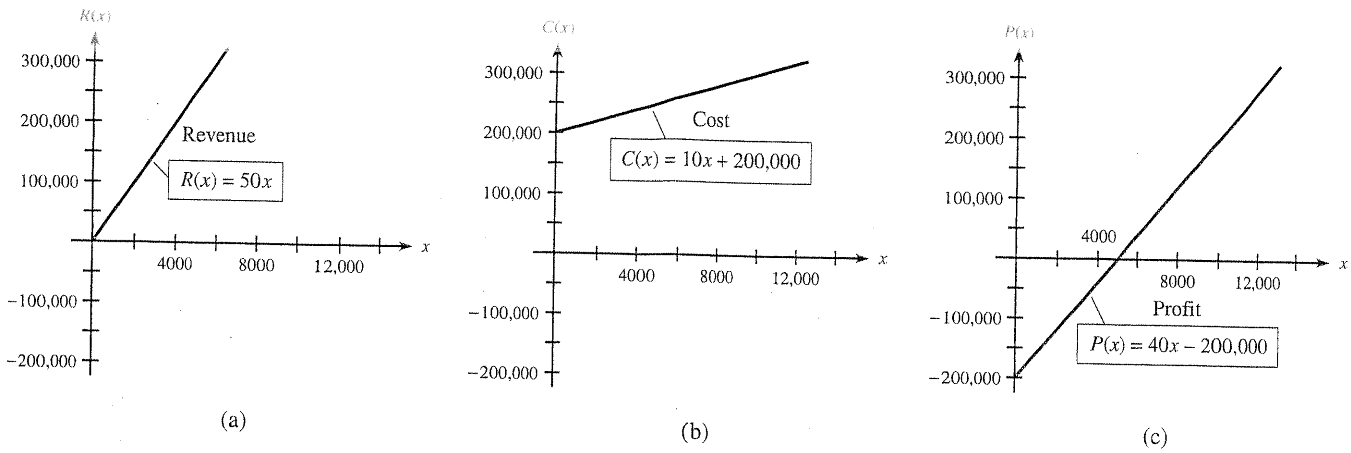


Figure 1.41

By observing the intercepts on the graphs in Figure 1.41, we note the following.

Revenue:	0 units produce 0 revenue.
Cost:	0 units' costs equal fixed costs = \$200,000.
Profit:	0 units yield a loss equal to fixed costs = \$200,000. 5000 units result in a profit of \$0 (no loss or gain).

Marginals

In Example 1, both the total revenue function and the total cost function are linear, so their difference, the profit function, is also linear. The slope of the profit function represents the rate of change in profit with respect to the number of units produced and sold. This is called the **marginal profit** ($\overline{MP}$) for the product. Thus the marginal profit for the MP3 players in Example 1 is \$40. Similarly, the **marginal cost** ($\overline{MC}$) for this product is \$10 (the slope of the cost function), and the **marginal revenue** ($\overline{MR}$) is \$50 (the slope of the revenue function).

EXAMPLE 2 *Marginal Cost*

Suppose that the cost (in dollars) for a product is $C = 21.75x + 4890$. What is the marginal cost for this product, and what does it mean?

Solution

The equation has the form $C = mx + b$, so the slope is 21.75. Thus the marginal cost is $\overline{MC} = 21.75$ dollars per unit.

Because the marginal cost is the slope of the cost line, production of each additional unit will cost \$21.75 more, at any level of production.

Note that when total cost functions are linear, the marginal cost is the same as the variable cost. This is not the case if the functions are not linear, as we shall see later.

Checkpoint

1. Suppose that when a company produces its product, fixed costs are \$12,500 and variable costs are \$75 per item.
   (a) Write the total cost function if x represents the number of units.
   (b) Are fixed costs equal to $C(0)$?
2. Suppose the company in Problem 1 sells its product for \$175 per item.
   (a) Write the total revenue function.
   (b) Find $R(100)$ and give its meaning.
3. (a) Give the formula for profit in terms of revenue and cost.
   (b) Find the profit function for the company in Problems 1 and 2.

● **EXAMPLE 3** ***Profit***

Suppose the profit function for a product is linear, and the marginal profit is \$5. If the profit is \$200 when 125 units are sold, write the equation of the profit function.

Solution

The marginal profit gives us the slope of the line representing the profit function. Using this slope ($m = 5$) and the point (125, 200) in the point-slope formula $P - P_1 = m(x - x_1)$ gives

$$P - 200 = 5(x - 125)$$

or

$$P = 5x - 425$$

Break-Even Analysis

We can solve the equations for total revenue and total cost simultaneously to find the point where cost and revenue are equal. This point is called the **break-even point.** On the graph of these functions, we use x to represent the quantity produced and y to represent the dollar value of revenue *and* cost. The point where the total revenue line crosses the total cost line is the break-even point.

● **EXAMPLE 4** ***Break Even***

A manufacturer sells a product for \$10 per unit. The manufacturer's fixed costs are \$1200 per month, and the variable costs are \$2.50 per unit. How many units must the manufacturer produce each month to break even?

Solution

The total revenue for x units of the product is $10x$, so the equation for total revenue is $R = 10x$. The fixed costs are \$1200, so the total cost for x units is $2.50x + 1200$. Thus the equation for total cost is $C = 2.50x + 1200$. We find the break-even point by solving the two equations simultaneously ($R = C$ at the break-even point). By substitution,

$$10x = 2.50x + 1200$$
$$7.5x = 1200$$
$$x = 160$$

Thus the manufacturer will break even if 160 units are produced per month. The manufacturer will make a profit if more than 160 units are produced. Figure 1.42 shows that for $x < 160$, $R(x) < C(x)$ (resulting in a loss) and that for $x > 160$, $R(x) > C(x)$ (resulting in a profit).

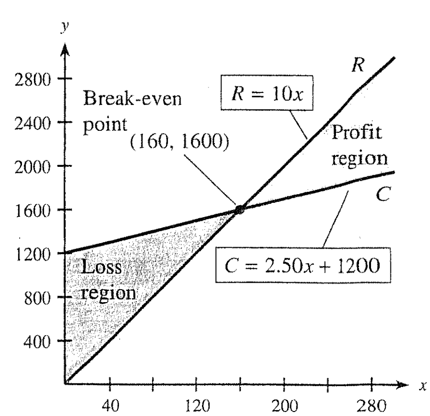


Figure 1.42

Using the fact that the profit function is found by subtracting the total cost function from the total revenue function, we can form the profit function for the previous example. The profit function is given by

$$P(x) = 10x - (2.50x + 1200) \quad \text{or} \quad P(x) = 7.50x - 1200$$

We can find the point where the profit is zero (the break-even point) by setting $P(x) = 0$ and solving for x.

$$0 = 7.50x - 1200$$
$$1200 = 7.50x$$
$$x = 160$$

Note that this is the same break-even point that we found by solving the total revenue and total cost equations simultaneously (see Figure 1.43).

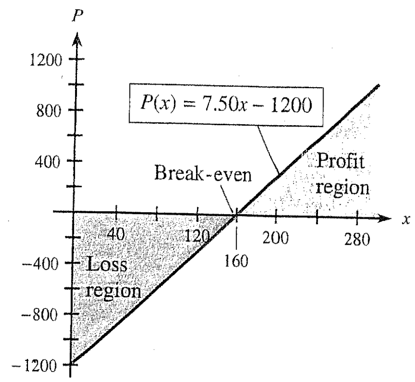


Figure 1.43

Checkpoint

4. Identify two ways in which break-even points can be found.

Supply, Demand, and Market Equilibrium

Economists and managers also use points of intersection to determine market equilibrium. **Market equilibrium** occurs when the quantity of a commodity demanded is equal to the quantity supplied.

Demand by consumers for a commodity is related to the price of the commodity. The **law of demand** states that the quantity demanded will increase as price decreases, and that the quantity demanded will decrease as price increases. Figure 1.44 shows the graph of a typical linear demand function. Note that although quantity demanded is a function of price, economists have traditionally graphed the demand function with price on the vertical axis. Throughout this text, we will follow this tradition. Linear equations relating price p and quantity demanded q can be solved for either p or q, and we will have occasion to use the equations in both forms.

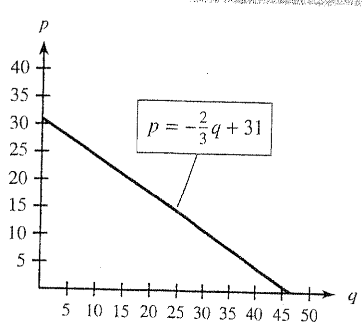


Figure 1.44

Just as a consumer's willingness to buy is related to price, a manufacturer's willingness to supply goods is also related to price. The **law of supply** states that the quantity supplied for sale will increase as the price of a product increases. Figure 1.45 shows the graph of a typical linear supply function. As with demand, price is placed on the vertical axis. Note that negative prices and quantities have no meaning, so supply and demand curves are restricted to the first quadrant.

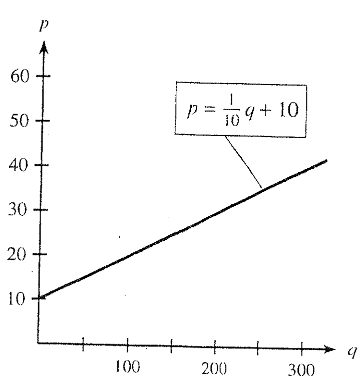


Figure 1.45

If the supply and demand curves for a commodity are graphed on the same coordinate system, with the same units, market equilibrium occurs at the point where the curves intersect. The price at that point is the **equilibrium price,** and the quantity at that point is the **equilibrium quantity.**

For the supply and demand functions shown in Figure 1.46, we see that the curves intersect at the point (30, 11). This means that when the price is \$11, consumers are willing to purchase the same number of units (30) that producers are willing to supply.

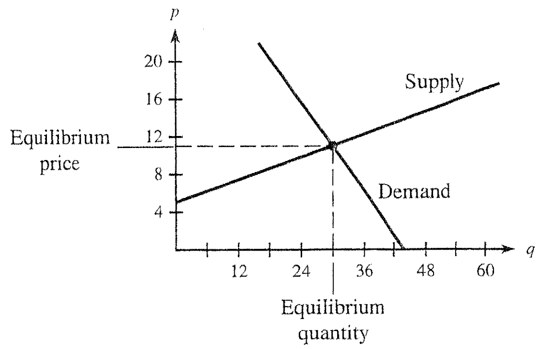


Figure 1.46

In general, the equilibrium price and the equilibrium quantity must both be positive for the market equilibrium to have meaning.

We can find the market equilibrium by graphing the supply and demand functions on the same coordinate system and observing their point of intersection. As we have seen, finding the point(s) common to the graphs of two (or more) functions is called **solving a system of equations** or **solving simultaneously.**

EXAMPLE 5 *Market Equilibrium*

Find the market equilibrium point for the following supply and demand functions.

$$\text{Demand:} \quad p = -3q + 36$$
$$\text{Supply:} \quad p = 4q + 1$$

Solution

At market equilibrium, the demand price equals the supply price. Thus,

$$\begin{aligned} -3q + 36 &= 4q + 1 \\ 35 &= 7q \\ q &= 5 \\ p &= 21 \end{aligned}$$

The equilibrium point is (5, 21).

Checking, we see that

$$21 = -3(5) + 36 \checkmark \quad \text{and} \quad 21 = 4(5) + 1 \checkmark$$

Spreadsheet Note

We can use Goal Seek with Excel to find the market equilibrium for given supply and demand functions. To find the equilibrium quantity and price for the supply and demand functions in Example 5, we set up the table with entries for quantity, demand, and supply. We enter the functions as in the following table and add a fourth entry representing demand − supply.

	A	B	C	D
1	q	p: demand	p: supply	demand − supply
2	1	=−3*A2+36	=4*A2+1	=B2−C2

We find the equilibrium quantity and price by using Goal Seek with D2 set to 0, with changing cell A2. The resulting solution gives the equilibrium quantity as 5 and the equilibrium price as 21. ■

	A	B	C	D
1	q	p: demand	p: supply	demand − supply
2	5	21	21	0

● **EXAMPLE 6** ***Market Equilibrium***

A group of wholesalers will buy 50 dryers per month if the price is \$200 and 30 per month if the price is \$300. The manufacturer is willing to supply 20 if the price is \$210 and 30 if the price is \$230. Assuming that the resulting supply and demand functions are linear, find the equilibrium point for the market.

Solution

Representing price by p and quantity by q, we have

Demand function:

$$m = \frac{300 - 200}{30 - 50} = -5$$
$$p - 200 = -5(q - 50)$$
$$p = -5q + 450$$

Supply function:

$$m = \frac{230 - 210}{30 - 20} = 2$$
$$p - 230 = 2(q - 30)$$
$$p = 2q + 170$$

Because the prices are equal at market equilibrium, we have

$$-5q + 450 = 2q + 170$$
$$280 = 7q$$
$$q = 40$$
$$p = 250$$

The equilibrium point is (40, 250). See Figure 1.47 for the graphs of these functions.

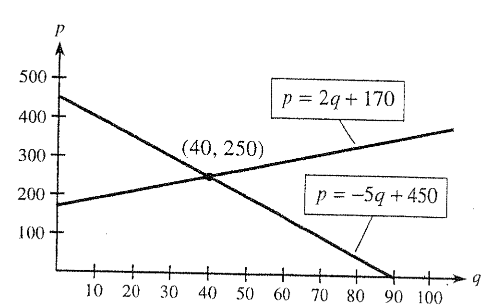


Figure 1.47

Supply and Demand with Taxation

Suppose a supplier is taxed \$$K$ per unit sold, and the tax is passed on to the consumer by adding \$$K$ to the selling price of the product. If the original supply function is $p = f(q)$, then passing the tax on gives a new supply function, $p = f(q) + K$. Because the value of the product is not changed by the tax, the demand function is unchanged. Figure 1.48 shows the effect that this has on market equilibrium.

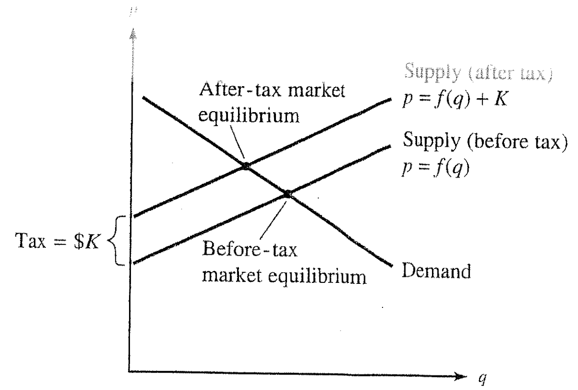


Figure 1.48

Note that the new market equilibrium point is the point of intersection of the original demand function and the new (after-tax) supply function.

● EXAMPLE 7 *Taxation*

In Example 6 the supply and demand functions for dryers were given as follows.

$$\text{Supply:} \quad p = 2q + 170$$
$$\text{Demand:} \quad p = -5q + 450$$

The equilibrium point was $q = 40$, $p = \$250$. If the wholesaler is taxed \$14 per unit sold, what is the new equilibrium point?

Solution

The \$14 tax per unit is passed on by the wholesaler, so the new supply function is

$$p = 2q + 170 + 14$$

and the demand function is unchanged. Thus we solve the system

$$\begin{cases} p = 2q + 184 \\ p = -5q + 450 \end{cases}$$
$$2q + 184 = -5q + 450$$
$$7q = 266$$
$$q = 38$$
$$p = 2(38) + 184 = 260$$

The new equilibrium point is $q = 38$, $p = \$260$.

Checking, we see that

$$260 = 2(38) + 184 \checkmark \quad \text{and} \quad 260 = -5(38) + 450 \checkmark$$

● Checkpoint

5. (a) Does a typical linear demand function have positive slope or negative slope? Why?
   (b) Does a typical linear supply function have positive slope or negative slope? Why?
6. (a) What do we call the point of intersection of a supply function and a demand function?
   (b) What algebraic technique is used to find the point named in (a)?

● Checkpoint Solutions

1. (a) $C(x) = 75x + 12{,}500$
   (b) Yes. $C(0) = 12{,}500$ = Fixed costs. In fact, fixed costs are defined to be $C(0)$.
2. (a) $R(x) = 175x$
   (b) $R(100) = 175(100) = \$17{,}500$, which means that revenue is \$17,500 when 100 units are sold.

3. (a) Profit = Revenue − Cost or $P(x) = R(x) - C(x)$
   (b) $P(x) = 175x - (75x + 12{,}500)$
   $= 175x - 75x - 12{,}500 = 100x - 12{,}500$
4. The break-even point occurs where revenue equals cost $[R(x) = C(x)]$ or where profit is zero $[P(x) = 0]$.
5. (a) Negative slope, because demand falls as price increases.
   (b) Positive slope, because supply increases as price increases.
6. (a) Market equilibrium
   (b) Solving simultaneously

1.6 Exercises

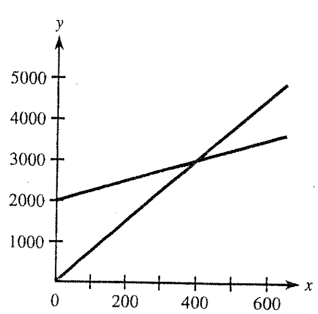
TOTAL COST, TOTAL REVENUE, AND PROFIT

1. Suppose a calculator manufacturer has the total cost function $C(x) = 17x + 3400$ and the total revenue function $R(x) = 34x$.
   (a) What is the equation of the profit function for the calculator?
   (b) What is the profit on 300 units?
2. Suppose a stereo receiver manufacturer has the total cost function $C(x) = 105x + 1650$ and the total revenue function $R(x) = 215x$.
   (a) What is the equation of the profit function for this commodity?
   (b) What is the profit on 50 items?
3. Suppose a radio manufacturer has the total cost function $C(x) = 43x + 1850$ and the total revenue function $R(x) = 80x$.
   (a) What is the equation of the profit function for this commodity?
   (b) What is the profit on 30 units? Interpret your result.
   (c) How many radios must be sold to avoid losing money?
4. Suppose a computer manufacturer has the total cost function $C(x) = 85x + 3300$ and the total revenue function $R(x) = 385x$.
   (a) What is the equation of the profit function for this commodity?
   (b) What is the profit on 351 items?
   (c) How many items must be sold to avoid losing money?
5. A linear cost function is $C(x) = 5x + 250$.
   (a) What are the slope and the C-intercept?
   (b) What is the marginal cost, and what does it mean?
   (c) How are your answers to (a) and to (b) related?
   (d) What is the cost of producing *one more* item if 50 are currently being produced? What is it if 100 are currently being produced?
6. A linear cost function is $C(x) = 27.55x + 5180$.
   (a) What are the slope and the C-intercept?
   (b) What is the marginal cost, and what does it mean?
   (c) How are your answers to (a) and to (b) related?
   (d) What is the cost of producing *one more* item if 50 are currently being produced? What is it if 100 are currently being produced?
7. A linear revenue function is $R = 27x$.
   (a) What is the slope?
   (b) What is the marginal revenue, and what does it mean?
   (c) What is the revenue received from selling *one more* item if 50 are currently being sold? If 100 are being sold?
8. A linear revenue function is $R = 38.95x$.
   (a) What is the slope?
   (b) What is the marginal revenue, and what does it mean?
   (c) What is the revenue received from selling *one more* item if 50 are currently being sold? If 100 are being sold?
9. Let $C(x) = 5x + 250$ and $R(x) = 27x$.
   (a) Write the profit function $P(x)$.
   (b) What is the slope of the profit function?
   (c) What is the marginal profit?
   (d) Interpret the marginal profit.
10. Given $C(x) = 21.95x + 1400$ and $R(x) = 20x$, find the profit function.
    (a) What is the marginal profit, and what does it mean?
    (b) What should a firm with these cost, revenue, and profit functions do? (*Hint:* Graph the profit function and see where it goes.)
11. A company charting its profits notices that the relationship between the number of units sold, x, and the profit, P, is linear. If 200 units sold results in \$3100 profit and 250 units sold results in \$6000 profit, write the profit function for this company. Find the marginal profit.

12. Suppose that the total cost function for a radio is linear, that the marginal cost is \$27, and that the total cost for 50 radios is \$4350. Write the equation of this cost function and then graph it.
13. Extreme Protection, Inc. manufactures helmets for skiing and snow boarding. The fixed costs for one model of helmet are \$6600 per month. Materials and labor for each helmet of this model are \$35, and the company sells this helmet to dealers for \$60 each.
    (a) For this helmet, write the function for monthly total costs.
    (b) Write the function for total revenue.
    (c) Write the function for profit.
    (d) Find $C(200)$, $R(200)$, and $P(200)$ and interpret each answer.
    (e) Find $C(300)$, $R(300)$, and $P(300)$ and interpret each answer.
    (f) Find the marginal profit and write a sentence that explains its meaning.
14. A manufacturer of DVD players has monthly fixed costs of \$9800 and variable costs of \$65 per unit for one particular model. The company sells this model to dealers for \$100 each.
    (a) For this model DVD player, write the function for monthly total costs.
    (b) Write the function for total revenue.
    (c) Write the function for profit.
    (d) Find $C(250)$, $R(250)$, and $P(250)$ and interpret each answer.
    (e) Find $C(400)$, $R(400)$, and $P(400)$ and interpret each answer.
    (f) Find the marginal profit and write a sentence that explains its meaning.

BREAK-EVEN ANALYSIS

15. The figure shows graphs of the total cost function and the total revenue function for a commodity.



    (a) Label each function correctly.
    (b) Determine the fixed costs.
    (c) Locate the break-even point and determine the number of units sold to break even.
    (d) Estimate the marginal cost and marginal revenue.

16. A manufacturer of shower-surrounds has a revenue function of

$$R(x) = 81.50x$$

and a cost function of

$$C(x) = 63x + 1850$$

Find the number of units that must be sold to break even.

17. A jewelry maker incurs costs for a necklace according to

$$C(x) = 35x + 1650$$

If the revenue function for the necklaces is

$$R(x) = 85x$$

how many necklaces must be sold to break even?

18. A small business recaps and sells tires. If a set of four tires has the revenue function

$$R(x) = 89x$$

and the cost function

$$C(x) = 1400 + 75x$$

find the number of sets of recaps that must be sold to break even.

19. A manufacturer sells belts for \$12 per unit. The fixed costs are \$1600 per month, and the variable costs are \$8 per unit.
    (a) Write the equations of the revenue and cost functions.
    (b) Find the break-even point.
20. A manufacturer sells watches for \$50 per unit. The fixed costs related to this product are \$10,000 per month, and the variable costs are \$30 per unit.
    (a) Write the equations of the revenue and cost functions.
    (b) How many watches must be sold to break even?
21. (a) Write the profit function for Problem 19.
    (b) Set profit equal to zero and solve for x. Compare this x-value with the break-even point from Problem 19(b).
22. (a) Write the profit function for Problem 20.
    (b) Set profit equal to zero and solve for x. Compare this x-value with the break-even point from Problem 20(b).
23. Electronic equipment manufacturer Dynamo Electric, Inc. makes several types of surge protectors. Their base model surge protector has monthly fixed costs of \$1045. This particular model wholesales for \$10 each and costs \$4.50 per unit to manufacture.
    (a) Write the function for Dynamo's monthly total costs.
    (b) Write the function for Dynamo's monthly total revenue.