LESSON 16.2 – MATRICES

$$\left[\begin{array}{cc|c} 6 & -1 & 3 \\ 4 & 4 & -1 \end{array}\right]$$

OVERVIEW

Here's what you'll learn in this lesson:

The Algebra of Matrices

a. Basic properties of matrices

b. Equality of matrices and addition, subtraction, and scalar multiplication of matrices

c. Product of matrices

d. The inverse of a matrix

The Gauss-Jordan Method

a. Linear systems and matrices

b. The Gauss–Jordan method

In this lesson, you will study matrices. First you'll learn about matrix addition, subtraction, and multiplication. Then you will use matrices to solve systems of linear equations. In particular, you will solve linear systems using the Gauss–Jordan Method.

EXPLAIN

THE ALGEBRA OF MATRICES

Summary

In this concept you will learn about arrays of numbers called matrices. You will learn to add, subtract, and multiply matrices.

Definition and Notation

A matrix is a rectangular array of numbers enclosed in brackets. A capital letter is usually used to name a matrix. Here are some examples of matrices.

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 2 & 6 \\ 5 & -8 & 11 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & -6 \\ 9 & -8 & 11 \\ -4 & 8 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 4 \\ 8 & 0 \\ 2 & 13 \\ 5 & -1 \\ 12 & 0 \end{bmatrix}$$

The size of a matrix is given by two numbers: the first number gives the number of rows of the matrix. The second number gives the number of columns of the matrix.

The matrix A is a 3 by 3 matrix. It has 3 rows and 3 columns.
The matrix B is a 4 by 3 matrix. It has 4 rows and 3 columns.
The matrix C is a 5 by 2 matrix. It has 5 rows and 2 columns.

You will often see "3 by 3" written as "3 x 3", "4 by 3" written as "4 x 3", "5 by 2" written as "5 x 2", and so on.

The numbers inside the brackets of a matrix are called its entries or elements. In a matrix A, these entries are identified using the notation a_{ij} where the subscript i describes the row number of the entry, and the subscript j describes the column number of the entry.

For a matrix B, you would use b_{ij} to denote its elements. For a matrix C, you would use c_{ij} to denote its elements, and so on.

So, for a 3 x 3 matrix A, you have $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Now look at the matrix $B = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 2 & 6 \\ 9 & -8 & 11 \\ -4 & 8 & -2 \end{bmatrix}$.

The entry $b_{13,}$ the element in Row 1 and Column 3, is 7.
The entry b_{32}, the element in Row 3 and Column 2, is −8.

Square Matrices

A matrix that has the same number of rows as columns is called a square matrix. Here are some examples of square matrices.

$$A = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 15 & 6 \\ 6 & -8 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 8 \\ -7 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 15 & 6 & 6 & -8 \\ 10 & 25 & -20 & 11 \\ -1 & 8 & 0 & -9 \end{bmatrix}$$

The size of A is 3 x 3. The size of B is 2 x 2. The size of C is 4 x 4.

A matrix that is not square does not have a main diagonal.

The main diagonal of a square matrix starts at the top left corner of the matrix and ends at the bottom right corner.

The elements of the main diagonal of the square matrix A are 1, 15 and 10.
The elements of the main diagonal of the square matrix B are 3 and 1.
The elements of the main diagonal of the square matrix C are 1, 6, –20 and –9.

Equality of Matrices

You'll now look at some operations on matrices. These are much like some operation on numbers.

Two matrices A and B are equal if they satisfy the following two conditions:

1. A and B are the same size.
2. Each entry in A is equal to its corresponding entry in B.

For example, if $A = \begin{bmatrix} 3 & \frac{1}{2} \\ 8 & 0 \\ 2.5 & 13 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & .5 \\ 8 & 0 \\ 2\frac{1}{2} & 13 \\ 5 & -1 \end{bmatrix}$, then A = B.

The matrices are equal because A and B are each 4 by 2 matrices, and each entry in A is equal to its corresponding entry in B.

You can write $\begin{bmatrix} 3 & \frac{1}{2} \\ 8 & 0 \\ 2.5 & 13 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 3 & .5 \\ 8 & 0 \\ 2\frac{1}{2} & 13 \\ 5 & -1 \end{bmatrix}$.

Here is an example of two matrices that are not equal.

$$\begin{bmatrix} 3 & 8 \\ -7 & 1 \end{bmatrix} \neq \begin{bmatrix} 3 & 8 \\ -7 & 1 \\ 4 & 0 \end{bmatrix}$$

Here, the first matrix is a 2 by 2 matrix and the second matrix is a 3 by 2 matrix.

Here is another example of two matrices that are not equal.

$$\begin{bmatrix} 3 & 8 \\ -7 & 1 \end{bmatrix} \neq \begin{bmatrix} 3 & 8 \\ -7 & 0 \end{bmatrix}$$

Here, both matrices are the same size. But, each of the corresponding entries is not equal. The entries in the second row, second column are different.

Addition and Subtraction of Matrices

Here's how to add (or subtract) two matrices:

1. Determine that the two matrices are the same size.
2. Add (or subtract) the corresponding entries.

For example, here's how to add the matrices $A = \begin{bmatrix} 3 & 8 \\ -7 & 1 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 7 \\ -7 & 0 \\ 5 & -3 \end{bmatrix}$:

1. The matrices are the same size.
   They are both 3 by 2.
2. Add the corresponding entries.

$$A + B = \begin{bmatrix} 3 & 8 \\ -7 & 1 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ -7 & 0 \\ 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 1 & 8 + 7 \\ -7 - 7 & 1 + 0 \\ 4 + 5 & 0 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 15 \\ -14 & 1 \\ 9 & -3 \end{bmatrix}$$

The matrices $\begin{bmatrix} 4 & 13 \\ -14 & 2 \\ 9 & -8 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 8 \\ -22 & 8 \end{bmatrix}$ *cannot be added or subtracted because they are not the same size.*

As another example, here's how to subtract the matrices $A = \begin{bmatrix} 1 & 8 \\ -6 & 1 \\ 3 & 0 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -5 & -3 \\ 6 & -2 \end{bmatrix}$:

1. The matrices are the same size. They are both 4 by 2.

2. Subtract the entries of the matrix B from the corresponding entries of the matrix A.

$$A - B = \begin{bmatrix} 1 & 8 \\ -6 & 1 \\ 3 & 0 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ -5 & -3 \\ 6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 8-2 \\ -6-2 & 1-0 \\ 3+5 & 0+3 \\ -1-6 & 2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 \\ -8 & 1 \\ 8 & 3 \\ -7 & 4 \end{bmatrix}$$

Zero Matrices

There is a special type of matrix called a zero matrix. It is any matrix all of whose entries are 0. You usually label a zero matrix as **0**.

A zero matrix can be any size.

Here are some examples of zero matrices.

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A zero matrix behaves much like the number zero. The number zero has the property that when you add zero to any number, you get back the number.

A zero matrix has the property that when it is added to any matrix A of the same size, you get back A.

That is, A + **0** = **0** + A = A.

For example, if $A = \begin{bmatrix} 4 & 13 \\ -14 & 2 \\ 9 & -8 \end{bmatrix}$ and $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, then A + **0** = A.

That is: $\begin{bmatrix} 4 & 13 \\ -14 & 2 \\ 9 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ -14 & 2 \\ 9 & -8 \end{bmatrix}$.

Similarly **0** + A = A.

Multiplication of a Matrix by a Real Number

You can multiply a matrix by a number.

To multiply a matrix, A, by a real number, *k*, (often called a scalar), multiply each entry of A by *k*. The new matrix is written *k*A or A*k*.

Here's an example of multiplying a matrix $A = \begin{bmatrix} 3 & 9 \\ -10 & 2 \\ 1 & -7 \end{bmatrix}$ by a scalar *k* = 5.

$$5A = 5\begin{bmatrix} 3 & 9 \\ -10 & 2 \\ 1 & -7 \end{bmatrix} = \begin{bmatrix} 5 \cdot 3 & 5 \cdot 9 \\ 5 \cdot (-10) & 5 \cdot 2 \\ 5 \cdot 1 & 5 \cdot (-7) \end{bmatrix} = \begin{bmatrix} 15 & 45 \\ -50 & 10 \\ 5 & -35 \end{bmatrix}$$

Here's another example. This time $A = \begin{bmatrix} 3 & 8 \\ -7 & 1 \end{bmatrix}$ and *k* = –2.

$$-2A = (-2)\begin{bmatrix} 3 & 8 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} -2 \cdot 3 & -2 \cdot 8 \\ -2 \cdot (-7) & -2 \cdot 1 \end{bmatrix} = \begin{bmatrix} -6 & -16 \\ 14 & -2 \end{bmatrix}$$

Multiplication of Two Matrices

Now you'll see how to multiply two matrices.

Here's an example.

If A = [2 4] and $B = \begin{bmatrix} 1 & 0 & -4 \\ 3 & -2 & 5 \end{bmatrix}$ you can find the matrix product AB like this:

The entry in Row 1 and Column 1 of AB is calculated as follows:

- Find Row 1 in the matrix A. $\quad$ [2 4]
- Find Column 1 in the matrix B. $\quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

- Multiply the corresponding elements and add the result.

$$[2\ \ 4]\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$= 2(1) + 4(3)$

$= 2 + 12$

$= 14$

The entry in Row 1 and Column 2 of AB is calculated as follows:

- Find Row 1 in the matrix A. $[2\ \ 4]$
- Find Column 2 in the matrix B. $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$
- Multiply the corresponding elements and add the result.

$$[2\ \ 4]\begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$= 2(0) + 4(-2)$

$= 0 - 8$

$= -8$

The entry in Row 1 and Column 3 of AB is calculated as follows:

- Find Row 1 in the matrix A. $[2\ \ 4]$
- Find Column 3 in the matrix B. $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$
- Multiply the corresponding elements and add the result.

$$[2\ \ 4]\begin{bmatrix} -4 \\ 5 \end{bmatrix}$$

$= 2(-4) + 4(5)$

$= -8 + 20$

$= 12$

The product AB is the matrix $[14\ \ -8\ \ 12]$.

So, $AB = [2\ \ 4] \cdot \begin{bmatrix} 1 & 0 & -4 \\ 3 & -2 & 5 \end{bmatrix} = [14\ \ -8\ \ 12]$.

Consider the size of the three matrices, A, B and AB.

$$\begin{array}{ccccc} A & \cdot & B & = & AB \end{array}$$

(1 by 2) · (2 by 3) = 1 by 3

You can find the matrix product, AB, because the number of columns, 2, of A is equal to the number of rows, 2, of B. The size of AB is given by the two outside numbers, 1 and 3. So, AB is 1 by 3.

Here are the steps to find the matrix product AB:

1. The number of columns of A must equal the number of rows of B.
2. In the matrix product AB, the entry in Row i and Column j, a_{ij}, is calculated as follows:
   - Find Row i in the matrix A.
   - Find Column j in the matrix B.
   - Multiply the corresponding elements and add the result.
3. Place each entry in the appropriate position in the matrix product AB.

Notice that if the size of the matrix A is m by n and the size of the matrix B is n by p, then the size of the matrix product AB is m by p.

So, for example, if you multiply a 3 x 5 matrix A and a 5 x 7 matrix B, then the product AB is a 3 x 7 matrix.

If A is a 2 x 3 matrix and B is a 4 x 5 matrix, then the product AB makes no sense. That's because 3, the number of columns of A, is not equal to 4, the number of rows of B.

Here's another example of a product AB.

$$A = \begin{bmatrix} 5 & 9 \\ -10 & 2 \\ 1 & -7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 8 & -7 & 1 \\ 1 & 4 & 0 & 2 \end{bmatrix}$$

The size of the matrix A is 3 by 2 and the size of the matrix B is 2 by 4. So the size of the matrix product AB is 3 by 4.

Here's how to find AB:

1. The number of columns of A is 2 which equals the number of rows of B.
2. In the matrix product AB, the entry in Row 1, Column 1 is calculated as follows:
   - Find Row 1 in the matrix A. $\begin{bmatrix} 5 & 9 \end{bmatrix}$
   - Find Column 1 in the matrix B. $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- Multiply the corresponding elements and add the result.

$$[5 \;\; 9]\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$= 5(3) + 9(1)$

$= 15 + 9$

$= 24$

3. Place this entry in Row 1, Column 1 of the matrix product, AB.

Now find the other entries of the matrix AB in a similar way.

Find Row 1, Column 2 of AB. $[5 \;\; 9]\begin{bmatrix} 8 \\ 4 \end{bmatrix} = 5(8) + 9(4) = 40 + 36 = 76$

Find Row 1, Column 3 of AB. $[5 \;\; 9]\begin{bmatrix} -7 \\ 0 \end{bmatrix} = 5(-7) + 9(0) = -35 + 0 = -35$

Find Row 1, Column 4 of AB. $[5 \;\; 9]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5(1) + 9(2) = 5 + 18 = 23$

Find Row 2, Column 1 of AB. $[-10 \;\; 2]\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-10)(3) + 2(1) = -30 + 2 = -28$

Find Row 2, Column 2 of AB. $[-10 \;\; 2]\begin{bmatrix} 8 \\ 4 \end{bmatrix} = (-10)(8) + 2(4) = -80 + 8 = -72$

Find Row 2, Column 3 of AB. $[-10 \;\; 2]\begin{bmatrix} -7 \\ 0 \end{bmatrix} = (-10)(-7) + 2(0) = 70 + 0 = 70$

Find Row 2, Column 4 of AB. $[-10 \;\; 2]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-10)(1) + 2(2) = -10 + 4 = -6$

Find Row 3, Column 1 of AB. $[1 \;\; -7]\begin{bmatrix} 3 \\ 1 \end{bmatrix} = (1)(3) + (-7)(1) = 3 - 7 = -4$

Find Row 3, Column 2 of AB. $[1 \;\; -7]\begin{bmatrix} 8 \\ 4 \end{bmatrix} = (1)(8) + (-7)(4) = 8 - 28 = -20$

Find Row 3, Column 3 of AB. $[1 \;\; -7]\begin{bmatrix} -7 \\ 0 \end{bmatrix} = (1)(-7) + (-7)(0) = -7 + 0 = -7$

Find Row 3, Column 4 of AB. $[1 \;\; -7]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1)(1) + (-7)(2) = 1 - 14 = -13$

So, $AB = \begin{bmatrix} 5 & 9 \\ -10 & 2 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 3 & 8 & -7 & 1 \\ 1 & 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 24 & 76 & -35 & 23 \\ -28 & -72 & 70 & -6 \\ -4 & -20 & -7 & -13 \end{bmatrix}$

Now here's an example where the matrix product AB does not make sense.

$A = \begin{bmatrix} 4 & 8 & -13 \\ 1 & 15 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 8 \\ -12 & 6 \end{bmatrix}$

Since A has 3 columns and B has 2 rows, the product AB is not defined.

In this example, you may notice that the product BA **does** make sense. That's because B has 2 columns and A has 2 rows. So the product BA is a 2 by 3 matrix.

B · A = BA

(2 by 2) · (2 by 3) = 2 by 3

You can find BA by following the steps for multiplication of two matrices. Here is the matrix product BA:

$BA = \begin{bmatrix} 1 & 8 \\ -12 & 6 \end{bmatrix} \begin{bmatrix} 4 & 8 & -13 \\ 1 & 15 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 128 & -13 \\ -42 & -6 & 156 \end{bmatrix}$

This example illustrates the fact that, for matrices A and B, in general, AB ≠ BA.

The Identity Matrix

You have already seen that when you are adding or subtracting matrices, the zero matrix **0** behaves like the number zero.

When you are multiplying square matrices, there is a matrix called the identity matrix, denoted with the letter **I**, that behaves like the number 1.

Here are some examples of identity matrices.

$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Identity matrices are square matrices with 1's on the main diagonal and 0's everywhere else. The identity matrix of size *n* by *n* has the property that when it is multiplied by any other matrix A of the same size, the result is A. The identity matrix behaves like the number one. That is, A**I** = **I**A = A.

Here is an example with $A = \begin{bmatrix} 1 & 7 \\ -13 & 6 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $A\mathbf{I} = \begin{bmatrix} 1 & 7 \\ -13 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1(1) + 7(0) & 1(0) + 7(1) \\ (-13)(1) + 6(0) & (-13)(0) + 6(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ -13 & 6 \end{bmatrix}$$

$$= A$$

And $\mathbf{I}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ -13 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 1(1) + 0(-13) & 1(7) + 0(6) \\ 0(1) + 1(-13) & 0(7) + 1(6) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ -13 & 6 \end{bmatrix}$$

$$= A$$

The Inverse of a Matrix

There is no division for matrices. You can, however, do something that resembles division, which is motivated by what you know about real numbers.

For example, $\frac{2}{7}$ is the multiplicative inverse of $\frac{7}{2}$ because $\frac{2}{7} \cdot \frac{7}{2} = 1$ and $\frac{7}{2} \cdot \frac{2}{7} = 1$.

Recall that the multiplicative inverse, $\frac{1}{a}$, of a non-zero number a is sometimes written a^{-1} and has the property that $aa^{-1} = a^{-1}a = 1$.

For some square matrices A, you can define the multiplicative inverse A^{-1} in the corresponding way:

$AA^{-1} = \mathbf{I}$ and $A^{-1}A = \mathbf{I}$

Here is an example of a square matrix, A, that has a multiplicative inverse, A^{-1}:

$$A = \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$$

You can check that $AA^{-1} = \mathbf{I}$ and $A^{-1}A = \mathbf{I}$.

$$AA^{-1} = \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 8(2) + 5(-3) & 8(-5) + 5(8) \\ 3(2) + 2(-3) & 3(-5) + 2(8) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \mathbf{I}$$

Similarly, $A^{-1}A = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix} \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 2(8) + (-5)(3) & 2(5) + (-5)(2) \\ -3(8) + 8(3) & -3(5) + 8(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \mathbf{I}$$

Finding the Inverse of a Matrix

You have just seen how to verify that a matrix is the multiplicative inverse of another matrix, by multiplying them together in either direction and checking that you get the identity matrix **I**. But how do you find the inverse of a given matrix?

Here is one way to find the inverse, if it exists, of a 2 by 2 square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. Find the determinant $|A| = \det A = ad - bc$.
2. Use the formula $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Since $|A|$ is in the denominator, assume that $|A| = ad - bc \neq 0$.

For example, let $A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

To find the inverse, A^{-1}, of A:

- Find the determinant $|A|$.

$$|A| = 3(2) - (-1)(-5)$$
$$= 6 - 5$$
$$= 1$$

- Use the formula.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{1}\begin{bmatrix} 2 & -(-1) \\ -(-5) & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

You can check that $AA^{-1} = A^{-1}A = \mathbf{I}$.

From this formula, you can see that, in general, if $|A| \neq 0$, then a 2 x 2 matrix A has an inverse, A^{-1}.

Answers to Sample Problems

Sample Problems

1a. What is the size of the matrix $A = \begin{bmatrix} 11 & 0 & 7 \\ 0 & -3 & 6 \\ 4 & -8 & 11 \\ -4 & 17 & -2 \end{bmatrix}$?

b. What is the entry a_{32}?

c. What entry a_{ij} is the number –2 ?

d. Does A have a main diagonal ?

☐ a. Find the number of rows.
Find the number of columns. ______ by ______

a. 4 by 3

✔ b. What row and column contain a_{32}? What number is there?
a_{32} describes the entry in Row 3 and Column 2. That number is –8.

☐ c. Find the row and column containing the entry –2. Identify the i and the j.
i = ______
j = ______

c. 4
3

☐ d. Does A have a main diagonal? ______________
Why or why not? ______________

d. No, because A is not a square matrix.

2. Given the matrices $A = \begin{bmatrix} 5 & -8 & 17 \\ 0 & 15 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 6 \\ 4 & -2 & -3 \end{bmatrix}$, calculate A – 3B.

✔ a. Use scalar multiplication to find the matrix 3B.

$$3B = 3\begin{bmatrix} 1 & 0 & 6 \\ 4 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 18 \\ 12 & -6 & -9 \end{bmatrix}$$

☐ b. Find the matrix A – 3B. A – 3B = ______________

b. $\begin{bmatrix} 2 & -8 & -1 \\ -12 & 21 & 5 \end{bmatrix}$

3. Given the matrices $A = \begin{bmatrix} -3 & 0 & 6 \\ 1 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 & -2 \\ -8 & -2 & 0 \\ -3 & -2 & -7 \end{bmatrix}$.

a. Does the matrix product AB exist?

b. If the product AB exists, what is its size?

c. If the product AB exists, find AB.

d. Does the matrix product BA exist?

☑ a. Check whether the number of columns of A equals the number of rows of B.

A has 3 columns.
B has 3 rows.
The number of columns of A equals the number of rows of B.
So, yes, the matrix product AB exists.

☐ b. Find the number of rows of A and the number of columns of B.

The size of AB is ________________ .

☐ c. Calculate the matrix product AB. AB = ________________

☐ d. Check whether the number of columns of B equals the number of rows of A.

____

4. Find the inverse of the matrix $A = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$.

Check your answer using multiplication.

☑ a. Find the determinant, $|A|$.

$$|A| = ad - bc$$
$$= 4(3) - 2(5)$$
$$= 12 - 10$$
$$= 2$$

☐ b. Use the formula to find A^{-1}.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

☐ c. Check that $AA^{-1} = \mathbf{I}$ and $A^{-1}A = \mathbf{I}$.

Answers to Sample Problems

b. 2 by 3

c. $\begin{bmatrix} -30 & -30 & -36 \\ 5 & 0 & -37 \end{bmatrix}$

d. The matrix product BA does not exist. The matrix B has 3 columns. The matrix A has 2 rows.

b. $\begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{bmatrix}$

c. $AA^{-1} = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4\left(\frac{3}{2}\right) + 2\left(-\frac{5}{2}\right) & 4(-1) + 2(2) \\ 5\left(\frac{3}{2}\right) + 3\left(-\frac{5}{2}\right) & 5(-1) + 3(2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$A^{-1}A = \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}(4) + (-1)(5) & \frac{3}{2}(2) + (-1)(3) \\ -\frac{5}{2}(4) + \quad 2(5) & -\frac{5}{2}(2) + \quad 2(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

THE GAUSS–JORDAN METHOD

Summary

You have already seen many applications which require you to find the solution of systems of linear equations. To solve such systems, you have used substitution, elimination, and determinants. Now, you will learn a new method. It uses matrices and is called the Gauss–Jordan Method.

Augmented Matrices

You can represent a system of linear equations using a matrix. Here are two examples.

The system of linear equations

$$3x - 5y = 17$$
$$14x + 7y = 23$$

can be represented using the augmented matrix $\left[\begin{array}{rr|r} 3 & -5 & 17 \\ 14 & 7 & 23 \end{array}\right]$.

The system of linear equations

$$20x - 4y + 5z = 1$$
$$3x + 11y - 8z = 20$$
$$5x - 9y - 13z = 13$$

can be represented using the augmented matrix $\left[\begin{array}{rrr|r} 20 & -4 & 5 & 1 \\ 3 & 11 & -8 & 20 \\ 5 & -9 & -13 & 13 \end{array}\right]$.

In each of these examples, the part of the matrix to the left of the solid line represents the coefficients of the system. The part of the matrix to the right of the solid line represents the constants of the system.

For instance, in the second example, the part of the matrix to the left of the solid line represents the coefficients of the variables x, y, and z. The part of the matrix to the right of the solid line represents the constants of the system.

You can also start with an augmented matrix and write the system of linear equations that it represents.

For example, the augmented matrix $\left[\begin{array}{rr|r} 13 & -8 & -7 \\ 6 & -8 & 15 \end{array}\right]$ represents the following system of linear equations in two variables:

Here, you have called the variables x and y.

$$13x - 8y = -7$$
$$6x - 8y = 15$$

Similarly, the augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 21 \end{array}\right]$ represents the following system of linear equations in 3 variables:

$x = 8$
$y = 13$
$z = 21$

Solving a System of Linear Equations using the Augmented Matrix

The left-hand column of the following chart shows an example of solving a system of linear equations by the elimination method. It motivates a general procedure that uses augmented matrices. As you look at each step in solving the system consider the corresponding step to the augmented matrix on the right:

Equations	**Augmented Matrix**
$x + y = -1$ $2x - 3y = 13$	$\left[\begin{array}{cc\|c} 1 & 1 & -1 \\ 2 & -3 & 13 \end{array}\right]$
Multiply the first equation by –2 and add the result to the second equation. $x + y = -1$ $0x - 5y = 15$	Multiply Row 1 by –2 and add the result to Row 2. $\left[\begin{array}{cc\|c} 1 & 1 & -1 \\ 0 & -5 & 15 \end{array}\right]$
Multiply the second equation by $-\frac{1}{5}$. $x + y = -1$ $y = -3$	Multiply Row 2 by $-\frac{1}{5}$. $\left[\begin{array}{cc\|c} 1 & 1 & -1 \\ 0 & 1 & -3 \end{array}\right]$
Multiply the second equation by –1 and add the result to the first equation. $x = 2$ $y = -3$	Multiply Row 2 by –1 and add the result to Row 1. $\left[\begin{array}{cc\|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$

The final augmented matrix has the identity matrix to the left of the solid line and the solution to the original system of linear equations to the right of the solid line.

The augmented matrix gives you a notation for recording the steps, using just the coefficients rather than the whole equations.

Elementary Row Operations

When you first learned to solve a linear equation there were several operations you were allowed to do to both sides of the equation. These operations gave you an equivalent equation that had the same solution as the original equation. Now you will see that similar operations can be used to transform an augmented matrix into an equivalent augmented matrix. These operations are called elementary row operations.

Operations on Equations	**Elementary Row Operations**
Interchange any two equations	Interchange any two rows
Multiply an equation by a nonzero constant	Multiply a row by a nonzero constant
Multiply an equation by a constant and add it to another equation	Multiply a row by a constant and add it to another row

Following are some examples of how an original matrix is transformed into an equivalent matrix using an elementary row operation.

Original Matrix	**Equivalent Matrix**	**Operation on Original Matrix**
$\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 2 & -3 & 13 \end{array}\right]$	$\left[\begin{array}{cc\|c} 5 & 10 & -5 \\ 2 & -3 & 13 \end{array}\right]$	Multiply Row 1 by 5.
$\left[\begin{array}{ccc\|c} 20 & -4 & 5 & 1 \\ 3 & 11 & -8 & 20 \\ 5 & -9 & -13 & 13 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 3 & 11 & -8 & 20 \\ 20 & -4 & 5 & 1 \\ 5 & -9 & -13 & 13 \end{array}\right]$	Interchange Row 1 and Row 2.
$\left[\begin{array}{ccc\|c} 20 & -4 & 5 & 1 \\ 3 & 11 & -8 & 20 \\ 5 & -9 & -13 & 13 \end{array}\right]$	$\left[\begin{array}{ccc\|c} 20 & -4 & 5 & 1 \\ 3 & 11 & -8 & 20 \\ 8 & 2 & -21 & 33 \end{array}\right]$	Multiply Row 2 by 1 and add to Row 3.
$\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 2 & -3 & 13 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & 2 & -1 \\ 5 & 3 & 10 \end{array}\right]$	Multiply Row 1 by 3 and add to Row 2.

The Gauss–Jordan Method

When you use elementary row operations to solve a system of linear equations, your goal is to transform the augmented matrix into an equivalent augmented matrix whose entries to the left of the solid line form an identity matrix. The numbers to the right of the solid line will then give the solution to the original system of linear equations.

For example, you previously saw this system of linear equations:

$$x + y = -1$$
$$2x - 3y = 13$$

When you transformed its augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & -3 & 13 \end{array}\right]$ into the augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$, the 2 by 2 identity matrix was to the left of the solid line and the solution to the original system of linear equations, $x = 2$ and $y = -3$, was to the right of the solid line.

There is a standardized sequence of steps which is frequently used to transform a given augmented matrix into an augmented matrix with an identity matrix to the left of the dotted line and the solution to the right of the dotted line. The method is called the Gauss–Jordan Method.

Here are the steps of the Gauss–Jordan Method that are used to transform an augmented matrix and solve a system of three linear equations in three variables.

1. Write the augmented matrix which corresponds to the system of linear equations.
2. Get a 1 in the upper left corner of the matrix.
3. In Column 1 of the matrix, get 0's below the 1.
4. Get a 1 in the a_{22} position of the matrix.
5. In Column 2 of the matrix, get 0's above and below this 1.
6. Get a 1 in the a_{33} position of the matrix.
7. In Column 3 of the matrix, get 0's above this 1.
8. Write the solution of the original system of linear equations.

Here's an example of how to solve the system of linear equations

$$x + y + z = 1$$
$$2x - y + z = 4$$
$$3x + 2y - 4z = 17$$

by using its corresponding augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 4 \\ 3 & 2 & -4 & 17 \end{array}\right]$.

Step	Matrix	Operation on previous matrix
1. Write the augmented matrix for the given system of linear equations.	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 4 \\ 3 & 2 & -4 & 17 \end{array}\right]$	
2. Get a 1 in the upper left corner (entry a_{11}).	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 4 \\ 3 & 2 & -4 & 17 \end{array}\right]$	a_{11} is already 1.
3. Get 0's below the 1 in Column 1.	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 3 & 2 & -4 & 17 \end{array}\right]$	Multiply Row 1 by –2 and add to Row 2.
	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & -7 & 14 \end{array}\right]$	Multiply Row 1 by –3 and add to Row 3.
4. Get a 1 in the a_{22} position.	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 1 \\ 0 & -1 & -7 & 14 \\ 0 & -3 & -1 & 2 \end{array}\right]$	Interchange Row 2 and Row 3.
	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 1 \\ 0 & 1 & 7 & -14 \\ 0 & -3 & -1 & 2 \end{array}\right]$	Multiply Row 2 by –1.
5. Get 0's above and below the 1 in the a_{22} position.	$\left[\begin{array}{rrr\|r} 1 & 0 & -6 & 15 \\ 0 & 1 & 7 & -14 \\ 0 & -3 & -1 & 2 \end{array}\right]$	Multiply Row 2 by –1 and add to Row 1.
	$\left[\begin{array}{rrr\|r} 1 & 0 & -6 & 15 \\ 0 & 1 & 7 & -14 \\ 0 & 0 & 20 & -40 \end{array}\right]$	Multiply Row 2 by 3 and add to Row 3.
6. Get a 1 in the a_{33} position.	$\left[\begin{array}{rrr\|r} 1 & 0 & -6 & 15 \\ 0 & 1 & 7 & -14 \\ 0 & 0 & 1 & -2 \end{array}\right]$	Multiply Row 3 by $\frac{1}{20}$.
7. In Column 3, get 0's above this 1.	$\left[\begin{array}{rrr\|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 7 & -14 \\ 0 & 0 & 1 & -2 \end{array}\right]$	Multiply Row 3 by 6 and add to Row 1.
	$\left[\begin{array}{rrr\|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$	Multiply Row 3 by –7 and add to Row 2.
8. Write the solution of the original system of linear equations.		$x = 3$ $y = 0$ $z = -2$

So the solution of the system of linear equations

$$\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= 4 \\ 3x + 2y - 4z &= 17 \end{aligned}$$

is $x = 3$
$y = 0$
$z = -2$.

As another example, here's a system of linear equations where you'll get fractions as you use the Gauss–Jordan Method to solve the system:

$$\begin{aligned} 2x + 8y &= 6 \\ 3x + y &= 5 \end{aligned}$$

Step	**Matrix**	**Operation on previous matrix**
1. Write the augmented matrix for the given system of linear equations.	$\left[\begin{array}{cc\|c} 2 & 8 & 6 \\ 3 & 1 & 5 \end{array}\right]$	
2. Get a 1 in the upper left corner (entry a_{11}).	$\left[\begin{array}{cc\|c} 1 & 4 & 3 \\ 3 & 1 & 5 \end{array}\right]$	Multiply Row 1 by $\frac{1}{2}$.
3. Get 0 below the 1 in Column 1.	$\left[\begin{array}{cc\|c} 1 & 4 & 3 \\ 0 & -11 & -4 \end{array}\right]$	Multiply Row 1 by –3 and add to Row 2.
4. Get a 1 in the a_{22} position.	$\left[\begin{array}{cc\|c} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{11} \end{array}\right]$	Multiply Row 2 by $-\frac{1}{11}$.
5. Get 0 above the 1 in the a_{22} position.	$\left[\begin{array}{cc\|c} 1 & 0 & \frac{17}{11} \\ 0 & 1 & \frac{4}{11} \end{array}\right]$	Multiply Row 2 by -4 and add to Row 1.
6. Write the solution of the original system of linear equations.		$x = \frac{17}{11}$ $y = \frac{4}{11}$

So the solution of the system of linear equations

$$\begin{aligned} 2x + 8y &= 6 \\ 3x + y &= 5 \end{aligned}$$

is $x = \frac{17}{11}$
$y = \frac{4}{11}$.

Dependent and Inconsistent Systems

Sometimes when you apply the steps for the Gauss–Jordan Method, you cannot get the identity matrix to the left of the solid line in the augmented matrix. There are two different situations that can cause this to occur.

1. A row of the augmented matrix is all zeros: $0\ 0\ 0\ |\ 0$

   This corresponds to the linear equation: $0x + 0y + 0z = 0$

In this situation, any values of x, y and z satisfy the equation.

When this occurs, the original system of linear equations is dependent, and it has an infinite number of solutions.

2. A row of the augmented matrix is all zeros except for the last entry which is nonzero. For example: $0\ 0\ 0\ |\ 7$

   This corresponds to the linear equation: $0x + 0y + 0z = 7$

In this situation, no values of x, y and z can satisfy this equation.

When this occurs, the original system of linear equations is inconsistent, and it has no solutions.

For systems of two variables you can easily use a graph to see the behavior described here. A dependent system of linear equations gives the same line twice when you draw the graphs. An inconsistent system of linear equations gives two parallel lines when you draw the graphs.

Here's an example of a dependent system of linear equations.

$$\begin{aligned} x + 3y &= 7 \\ 2x + 6y &= 14 \end{aligned}$$

Notice what happens when you try to solve the system using the Gauss–Jordan Method.

Step	**Matrix**	**Operation on original matrix**
1. Write the augmented matrix for the given system of linear equations.	$\left[\begin{array}{cc\|c} 1 & 3 & 7 \\ 2 & 6 & 14 \end{array}\right]$	
2. Get 0's below the 1 in Column 1.	$\left[\begin{array}{cc\|c} 1 & 3 & 7 \\ 0 & 0 & 0 \end{array}\right]$	Multiply Row 1 by –2 and add to Row 2.
3. This is a dependent system.		The last row is all zeros.

The original system of linear equations has an infinite number of solutions.

Here's an example of an inconsistent system of linear equations.

$$\begin{aligned} x + 3y &= 7 \\ 2x + 6y &= 20 \end{aligned}$$

Notice what happens when you try to solve the system using the Gauss–Jordan Method.

Step	**Matrix**	**Operation on original matrix**
1. Write the augmented matrix corresponding to the system of linear equations.	$\left[\begin{array}{cc\|c} 1 & 3 & 7 \\ 2 & 6 & 20 \end{array}\right]$	
2. Get 0's below the 1 in Column 1.	$\left[\begin{array}{cc\|c} 1 & 3 & 7 \\ 0 & 0 & 6 \end{array}\right]$	Multiply Row 1 by –2 and add to Row 2.
3. This is an inconsistent system.		The last row is 0 0 | 6.

The original system of linear equations has no solutions.

Answers to Sample Problems

b. $x + y + 2z = 1$
$3y + 23z = -33$
$16z = -48$

a. A and C

Sample Problems

1. a. Write the augmented matrix that corresponds to the following system of linear equations:

$$\begin{aligned} 3x - 4y + z &= 12 \\ 5x + y + 2z &= 4 \\ 2x - 7y - 8z &= 1 \end{aligned}$$

b. Write the system of linear equations that corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 3 & 23 & -33 \\ 0 & 0 & 16 & -48 \end{array}\right]$$

✔ a. Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & -4 & 1 & 12 \\ 5 & 1 & 2 & 4 \\ 2 & -7 & -8 & 1 \end{array}\right]$$

☐ b. Write the system of linear equations. ____________________
____
____

2. a. Which two matrices below are equivalent? Remember, two matrices are equivalent if either matrix can be transformed into the other matrix by one or more elementary row operations. (In this example, only one elementary row operation has been used.)

$$A = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 15 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 0 & -24 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 6 \\ 2 & 15 & 8 \end{bmatrix}$$

b. Describe the row operation used to transform one matrix from a) into the other matrix.

☐ a. Which two matrices are equivalent? ____________________

✔ b. Describe the row operation. Multiply Row 1 of A by 2, and add the result to Row 2.

3. Solve the following system of linear equations using the Gauss–Jordan Method.

$3x + 6y = -12$
$2x + 5y = 7$

✔ a. Write the augmented matrix corresponding to the system.

$$\left[\begin{array}{cc|c} 3 & 6 & -12 \\ 2 & 5 & 7 \end{array}\right]$$

☐ b. Get a 1 in the upper left corner (entry a_{11}). ________________

☐ c. Get 0 below the 1 in Column 1. ________________

☐ d. Get a 1 in the a_{22} position. ________________

☐ e. Get 0 above this 1 in Column 2. ________________

☐ f. Write the solution of the original system of linear equations. ________________

4. From the augmented matrices below, pick one that represents an inconsistent system. Then pick one that represents a dependent system.

$$A = \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & 9 & 32 \end{array}\right] \quad B = \left[\begin{array}{cc|c} 2 & 5 & 6 \\ 0 & 0 & 22 \end{array}\right] \quad C = \left[\begin{array}{cc|c} 1 & 4 & 7 \\ 2 & 8 & 14 \end{array}\right]$$

✔ a. Find an inconsistent system.

The matrix B represents an inconsistent system. The last row is 0 0 | 22, which represents the equation $0x + 0y = 22$, an impossibility.

☐ b. Find a dependent system. ________________

Answers to Sample Problems

b. Multiply Row 1 by $\frac{1}{3}$.

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 2 & 5 & 7 \end{array}\right]$$

c. Multiply Row 1 by –2 and add the result to Row 2.

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 1 & 15 \end{array}\right]$$

d. a_{22} is already equal to 1.

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & 1 & 15 \end{array}\right]$$

e. Multiply Row 2 by –2 and add the result to Row 1.

$$\left[\begin{array}{cc|c} 1 & 0 & -34 \\ 0 & 1 & 15 \end{array}\right]$$

f. $x = -34$
$y = 15$

b. The matrix C represents a dependent system. That's because –2 times Row 1 added to Row 2 transforms the last row to 0 0 | 0, which represents the equation $0x + 0y = 0$. This equation has an infinite number of solutions.

HOMEWORK

Homework Problems

Circle the homework problems assigned to you by the computer, then complete them below.

Explain

The Algebra of Matrices

1. Write the size of each matrix.

$$A = \begin{bmatrix} 14 & 82 & -13 \\ 1 & -15 & 10 \end{bmatrix} \quad B = \begin{bmatrix} 8 \\ 3 \\ 11 \\ 12 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 \\ 7 & 13 \end{bmatrix}$$

2. Given the matrix $A = \begin{bmatrix} 3 & -4 \\ 7 & 0 \\ 2 & 11 \\ 5 & -1 \\ -12 & 0 \end{bmatrix}$, find the matrix –2A.

3. Given the matrices $A = \begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 4 & 7 \end{bmatrix}$,

   calculate the matrix products AB and BA.

4. Let the matrix $A = \begin{bmatrix} 1 & -2 & 7 & 10 \\ 15 & 6 & -6 & -8 \\ 10 & 24 & -20 & 11 \\ -3 & 28 & 0 & -19 \end{bmatrix}$.

   a. What is the entry a_{13}?

   b. What is the entry a_{34}?

   c. What entry is the number 28?

5. Given the matrices $A = \begin{bmatrix} 3 & 8 \\ -17 & 2 \\ 14 & -7 \end{bmatrix}$, $B = \begin{bmatrix} 21 & 1 \\ 8 & 3 \end{bmatrix}$,

   and $C = \begin{bmatrix} 3 & 23 \\ -18 & 2 \\ 6 & -7 \end{bmatrix}$, which of the following operations are possible?

   A + B   C – A   B + 2C   A – 3C

6. Given the matrices $M = \begin{bmatrix} 1 & 8 & -3 \\ 1 & 4 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & 0 \\ -2 & 2 \\ 4 & -6 \end{bmatrix}$, calculate MN.

7. Which of the matrices below does not have a main diagonal?

   For the other matrices, write down the elements of the main diagonal.

$$A = \begin{bmatrix} 3 & 9 \\ -10 & 2 \\ 1 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 11 & 7 & -9 \\ 5 & -3 & 0 \\ 8 & 0 & -25 \end{bmatrix}$$

8. Given the matrices $A = \begin{bmatrix} 2 & 1 \\ 8 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -7 & 4 \end{bmatrix}$, calculate A + 2B.

9. Use multiplication to check that the matrix $\begin{bmatrix} \frac{3}{2} & -2 \\ 2 & -3 \end{bmatrix}$ is the inverse of the matrix $\begin{bmatrix} 6 & -4 \\ 4 & -3 \end{bmatrix}$.

10. Which of the following matrices are equal to each other?

$$A = \begin{bmatrix} .5 & 8 & -13 \\ -\frac{3}{2} & 15 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 8 & -14 \\ -1.5 & 15 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \frac{1}{2} & 8 & -13 \\ -1.5 & 15 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 8 \\ -1.5 & 15 \end{bmatrix}$$

11. Given the matrices $M = \begin{bmatrix} .5 & 1 & -12 \\ -3 & 9 & 5 \end{bmatrix}$

    and $N = \begin{bmatrix} 2 & -8 & 7 \\ -1 & 0 & 10 \end{bmatrix}$, calculate 2M – 3N.

12. Find the inverse of the matrix $A = \begin{bmatrix} 5 & 4 \\ -3 & -3 \end{bmatrix}$.

The Gauss–Jordan Method

13. Write the augmented matrix for the following system of linear equations:

$$\begin{aligned} 11x - y &= 34 \\ 2x + 8y &= 4 \end{aligned}$$

14. Given the augmented matrix $\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 8 & -1 & 7 \end{array}\right]$, find the matrix you obtain when you multiply Row 1 by -4 and add the result to Row 2.

15. Use the Gauss–Jordan Method to solve the following system of linear equations:

$$\begin{aligned} x + y &= 4 \\ 2x - y &= 5 \end{aligned}$$

16. Write the augmented matrix for the following system of linear equations:

$$\begin{aligned} 5x - y + 2z &= -7 \\ 4x + 11y - 13z &= 20 \\ x - 8y - 12z &= 5 \end{aligned}$$

17. For a given matrix, which of the following operations are elementary row operations?

    a. Multiply each entry of Row 1 by the corresponding entry of Row 2.

    b. Multiply the entries of some row by 1 and add the result to another row.

    c. Multiply every entry in a row by 5.

    d. Take the square root of each entry in a row.

18. Use the Gauss–Jordan Method to solve the following system of linear equations:

$$\begin{aligned} x + y + z &= 4 \\ 2x - y + 3z &= 15 \\ 2x + 2y - z &= -4 \end{aligned}$$

19. Write the system of linear equations that corresponds to this augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -18 \end{array}\right]$$

20. Given the augmented matrix $\left[\begin{array}{ccc|c} 0 & -1 & 2 & -3 \\ 4 & 0 & -1 & 2 \\ 1 & -8 & -2 & 5 \end{array}\right]$, find the matrix you get by multiplying Row 1 by –2 and adding the result to Row 2.

21. In using the Gauss–Jordan Method to solve the following system of linear equations, what can you conclude?

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + 3z &= 7 \\ 4x + y + 5z &= 19 \end{aligned}$$

22. Write the system of linear equations that corresponds to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & -1 & 5 & -8 \\ -4 & 13 & -20 & 8 \\ -1 & -7 & -12 & 4 \end{array}\right]$$

23. Find which of the following matrices are equivalent. (Hint: use elementary row operations.)

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 8 & 15 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$$

24. Which augmented matrix below corresponds to a dependent system?
Which one corresponds to an inconsistent system?

$$A = \left[\begin{array}{cc|c} 3 & 1 & 7 \\ 6 & 2 & 14 \end{array}\right] \qquad B = \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 4 & 1 & -18 \end{array}\right]$$

$$C = \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 3 & 6 & 5 \end{array}\right]$$

APPLY

Practice Problems

Here are some additional practice problems for you to try.

The Algebra of Matrices

1. Find the size of the matrices $A = \begin{bmatrix} 4 & 4 \\ 8 & 0 \\ -2 & 13 \\ -5 & -8 \\ 17 & 0 \\ 11 & -34 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -8 & -19 \\ 11 & 16 & 0 \end{bmatrix}$.

2. Find the following entries in the matrix
$$A = \begin{bmatrix} -21 & 0 & 8 \\ 0 & -18 & 6 \\ 43 & -8 & 13 \\ -4 & 27 & -12 \end{bmatrix}.$$
a_{13}
a_{42}
a_{22}

3. Find the entries of the main diagonal of the following matrices, if the main diagonal exists.
$$A = \begin{bmatrix} 4 & 8 & -13 & 1 & 15 \\ 0 & 7 & -1 & 23 & -9 \end{bmatrix} \quad B = \begin{bmatrix} 100 & 57 \\ -18 & 31 \end{bmatrix}$$

4. In the following list, find any matrices that are equal:
$$A = \begin{bmatrix} 10 & 0.5 & -1 \\ \frac{3}{2} & 6 & 0 \end{bmatrix} \quad I = \begin{bmatrix} 10 & \frac{1}{2} \\ -1 & 1.5 \\ 6 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 10 & \frac{1}{2} & -1 \\ 1.5 & 6 & 0 \end{bmatrix}$$

5. In the following list, find a zero matrix and an identity matrix:
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Given the matrices $A = \begin{bmatrix} 1 & 9 \\ -10 & 2 \\ -1 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 4 \\ -9 & -2 \\ 11 & -7 \end{bmatrix}$, find the matrix A – B.

7. Given the matrices $A = \begin{bmatrix} 1 & 9 \\ -10 & 2 \\ -1 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} -6 & 4 \\ -9 & -2 \\ 11 & -7 \end{bmatrix}$, find the matrix A + 2B.

8. Given the matrices $A = \begin{bmatrix} 3 & -1 & -3 \\ 2 & 0 & -6 \\ 5 & -2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & -4 \\ 1 & 6 & -1 \end{bmatrix}$, find the matrix product AB.

9. Given the matrices $A = \begin{bmatrix} 2 & 5 & -1 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -5 & -2 & 1 \\ 3 & 3 & 0 & 1 \\ 2 & 1 & 3 & -2 \\ 4 & 1 & 7 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix}$ find the matrix product AB.

   What can you say about the matrix product BA?

10. Find the inverse of the matrix $A = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$.

The Gauss-Jordan Method

11. Write the augmented matrix for this system of linear equations:
$$2x - 3y = 7$$
$$5x + 11y = 8$$

12. Write a system of linear equations that corresponds to this augmented matrix:
$$\left[\begin{array}{ccc|c} 24 & -13 & 7 & 2 \\ -3 & 12 & -9 & 13 \\ -5 & -8 & 10 & -4 \end{array}\right]$$

13. Write the matrix you get when you switch row 1 and row 2 of this augmented matrix:

$$\left[\begin{array}{cc|c} 6 & -1 & 3 \\ 4 & 2 & -1 \end{array}\right]$$

14. Write the matrix you get when you multiply row 2 by the constant –1 and add the result to row 3 of this augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & -9 & 0 & 2 \\ -1 & 4 & -9 & 3 \\ -5 & -7 & 8 & -4 \end{array}\right]$$

15. Describe the row operation that was used to transform the augmented matrix $\left[\begin{array}{cc|c} 3 & -6 & 9 \\ -7 & -8 & -3 \end{array}\right]$ into the augmented matrix $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ -7 & -8 & -3 \end{array}\right]$.

16. Use the Gauss-Jordan Method to solve this system of linear equations:

$$\begin{aligned} x + y &= 8 \\ 2y &= 10 \end{aligned}$$

17. Use the Gauss-Jordan Method to solve this system of linear equations:

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + z &= 11 \\ y - z &= 5 \end{aligned}$$

18. Use the Gauss-Jordan Method to solve this system of linear equations:

$$\begin{aligned} 2x + 3y &= 7 \\ 3x - 2y &= 17 \end{aligned}$$

19. Use the Gauss-Jordan Method to identify which of the following linear systems is inconsistent:

$$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned} \qquad \begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 7 \end{aligned}$$

20. Use the Gauss-Jordan Method to identify which of the following linear systems is dependent:

$$\begin{aligned} 3x - y + z &= 1 \\ 2x + 3y + 4z &= 7 \\ 10x + 4y + 10z &= 16 \end{aligned} \qquad \begin{aligned} 3x - y + z &= 1 \\ 2x + 3y + 4z &= 7 \\ 15x + 6y + 15z &= 16 \end{aligned}$$

EVALUATE

Practice Test

Take this practice test to be sure that you are prepared for the final quiz in Evaluate.

1. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 11 & 12 \\ 15 & 7 & -4 & -8 \\ 10 & 34 & -25 & 11 \\ -3 & 23 & 0 & -18 \\ 1 & -3 & 17 & 55 \end{bmatrix}$.

    a. What is the size of the matrix A?

    b. Find the entries a_{23} and a_{41}.

2. Given the matrices $A = \begin{bmatrix} -5 & 10 & 3 \\ -3 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 2 & -4 \\ -1 & 1 & 0 \end{bmatrix}$, find the matrix 2A – 3B.

3. Given the matrices $A = \begin{bmatrix} -1 & 8 & 3 \\ 0 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -4 \\ -2 & 1 & 0 \\ 3 & -5 & 2 \end{bmatrix}$, find the matrix product AB.

4. Determine which matrix below is the multiplicative inverse of the matrix $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$.

    $B = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$ $\qquad$ $C = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

5. Write the augmented matrix that corresponds to the following system of linear equations:

$$\begin{aligned} -3x + 4y - 7z &= 12 \\ 4x - 8y - 11z &= 28 \\ x - 13y + 5z &= -18 \end{aligned}$$

6. Perform the following row operation on the matrix A:

    Multiply Row 2 by –3 and add the result to Row 3.

$$A = \left[\begin{array}{ccc|c} 5 & 7 & -14 & 23 \\ -1 & -13 & 5 & -8 \\ -3 & 1 & -6 & -2 \end{array}\right]$$

7. Use the Gauss–Jordan Method to solve the following system of linear equations:

$$\begin{aligned} x + y + z &= 6 \\ 3y - 3z &= -18 \\ 2y + z &= 3 \end{aligned}$$

8. Identify which of the two examples below represents a dependent system of linear equations and which represents an inconsistent system equations.

    a. $\begin{aligned} x + 2y &= 5 \\ 3x + 6y &= 8 \end{aligned}$

    b. $\begin{aligned} x - y - z &= 5 \\ 2x + y + 3z &= 8 \\ 3x + 2z &= 13 \end{aligned}$

TOPIC 16 CUMULATIVE ACTIVITIES

CUMULATIVE REVIEW PROBLEMS

These problems combine all of the material you have covered so far in this course. You may want to test your understanding of this material before you move on to the next topic. Or you may wish to do these problems to review for a test.

1. Solve by using the quadratic formula: $x^2 + 3x - 3 = 0$.

2. Find: $(2x + \sqrt{5y})(3x - 3\sqrt{5y})$

3. Find: $\frac{3}{-4} \cdot \left(\frac{-14}{5}\right) \div \frac{21}{10}$

4. Find in standard form the equation of the ellipse with center at the origin, major axis of length 10 along the x-axis, and minor axis of length 6 along the y-axis.

5. Factor: $15a^3b^5 - 5a^4b^4 + 30a^5$

6. Find the x-intercepts and the y-intercept of the function $y = f(x) = |x - 2| - 4$. Graph the function.

7. Given the matrices $A = \begin{bmatrix} 2 & 8 & -4 \\ 0 & -3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 10 & -1 & -9 \\ 6 & 1 & -7 \end{bmatrix}$, find the matrix 3A – 2B.

8. Reduce to lowest terms: $\frac{2x^2 - 50}{2x + 10}$

9. One of the terms of the partial fraction decomposition of $\frac{5x}{x^2 - x - 6}$ is $\frac{2}{x + 2}$. Find the other term.

10. Use the Gauss–Jordan Method to solve the following system of equations:

$$\begin{aligned} 2x + 3y - 4z &= 6 \\ x + 2y - 3z &= 3 \\ -2x - z &= -5 \end{aligned}$$

11. Find: $3 \cdot (-4) + (-24) \div (-4)$

12. Simplify: $\frac{1 + 2i}{1 - 2i}$

13. If $P(x) = x^3 - 2ix^2 + 4x - 8i$, find $P(2i)$.

14. Find the determinant of the matrix A.

$$A = \begin{bmatrix} -3 & 2 \\ -4 & 5 \end{bmatrix}$$

15. For the function $f(x) = x^3 - 2x^2 + 4$, find:

    a. $f(-2)$

    b. $f(2x)$

16. Find: $\frac{24y^2 - 18y^5 + 12y^3 - 36}{6y^3}$

17. Find the inverse of the function $f(x) = 4x + 2$.

18. Suppose you are using Cramer's rule to solve a system of two linear equations. You find that the determinant of the coefficient matrix A is zero. Circle the possible types of solutions of the system.

    a. The system has a unique solution.

    b. The system has no solution.

    c. The system has an infinite number of solutions.

19. Find the equation of the parabola that opens to the right, has vertex (2, 5), and goes through the point (3, 1).

20. Solve for t: $5^{t+1} = 25^{t-1}$

21. Find the center of the ellipse whose equation is $\frac{(y + 3)^2}{81} + \frac{(x - 3)^2}{49} = 1$.

22. Use properties of logarithms to rewrite this expression as a single logarithm:

$$\ln x^2 - \ln (x + 2) + \ln (x - 2)$$

23. Graph the hyperbola whose equation is $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

24. Use your calculator and the change of base formula to approximate $\log_3 44.5$ to two decimal places.

25. Which of the polynomial functions below are symmetric about the y-axis?

a. $y = P(x) = 2x^3 + x + 1$

b. $y = P(x) = -2x^2 + 1$

c. $y = P(x) = 2x^2 + 2x + 1$

d. $y = P(x) = 2x^4 + 3x^2$

26. The graph of $y = \frac{1}{4}x^2$ and the graph of another function are shown in Figure 16.1. What is the other function?

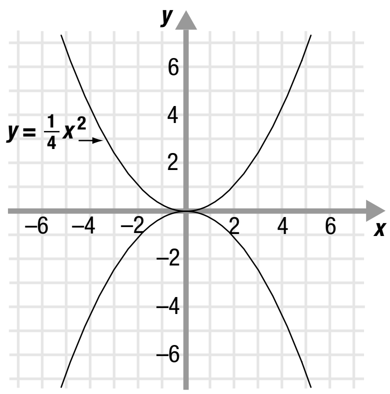

Figure 16.1

27. Find two values of x that are not in the domain of this function:

$f(x) = \dfrac{x+7}{x^2 - 5x - 6}$

28. Find the horizontal asymptote for the function $y = H(r) = \dfrac{-4r^2 + 8}{r^2 - 8}$.

29. Find the vertical asymptote and the oblique asymptote for the function $y = f(x) = \dfrac{3x^2 + 8x - 4}{x - 2}$.

30. Solve this inequality for x: $|4x + 4| \geq 8$

31. Given that $Q(x) = 6x^3 - 32x^2 + 15x - 5$, use synthetic division and the Remainder Theorem to find Q(5).

32. Is $x + 3$ a factor of $Q(x) = x^4 + x^3 - 6x^2 + x + 3$?

33. Given the matrices $A = \begin{bmatrix} 1 & 6 \\ -2 & 3 \end{bmatrix}$,

and $B = \begin{bmatrix} -2 & -1 & 5 \\ 0 & -4 & 2 \end{bmatrix}$, find the matrix product AB.

34. Given that $P(x) = x^3 - 3x^2 - 3x - 4$ and $P(4) = 0$, find two factors of $P(x)$.

35. Given that Q(x) is a polynomial with real coefficients and that $-3 - 3i$ is a factor of Q(x), find another factor of Q(x).

36. Use the Intermediate Value Theorem to determine if $P(x) = x^4 - x^2 + 3x - 2$ has a possible zero between 0 and 1.

37. Factor: $3x^4 - 48y^{12}$

38. The equation of a circle is $x^2 + 2x + y^2 - 4y = 11$. Find the radius of the circle.

39. Solve for x: $\left(\frac{1}{4}\right)^{2x-1} = 64$

40. Simplify using properties of exponents: $\dfrac{2^7 \cdot z^6}{2^6 \cdot z^7}$

41. Write this exponential statement in logarithmic form: $7^{-2x} = 99$

42. Graph the parabola whose equation is $(y - 1)^2 = 4(x + 2)$.

43. Solve for x: $8 + 4\log_4 (x + 1) = 24$

44. Find the equation of the line perpendicular to $x + y = 5$ that passes through the point (–2, –1). Write your answer in slope-intercept form.

45. Find the element, d, of the matrix A that makes det A = 0.

$A = \begin{bmatrix} 4 & -3 \\ d & -6 \end{bmatrix}$

46. Find the distance between the points (–2, 7) and (–8, –1).

47. Here is a system of linear equations:

$$3x + 4y = 8$$
$$2x - 7y = 9$$

Write the coefficient matrix, A, of this system.

48. Given the functions $f(x) = 2x^2 + 7x - 3$ and $g(x) = 3x^2 - x$, find $\frac{f}{g}(-1)$.

49. Solve this equation for x: $|x + 7| = 18$

50. Circle all the terms below that are included in the form of the partial fraction decomposition of $\dfrac{4x^2 + 7x - 6}{(x-1)(x+4)^2}$.

a. $\dfrac{Ax + B}{x^2 + 3x - 4}$

b. $\dfrac{A}{x - 1}$

c. $\dfrac{B}{x + 4}$

d. $\dfrac{C}{(x + 4)^2}$