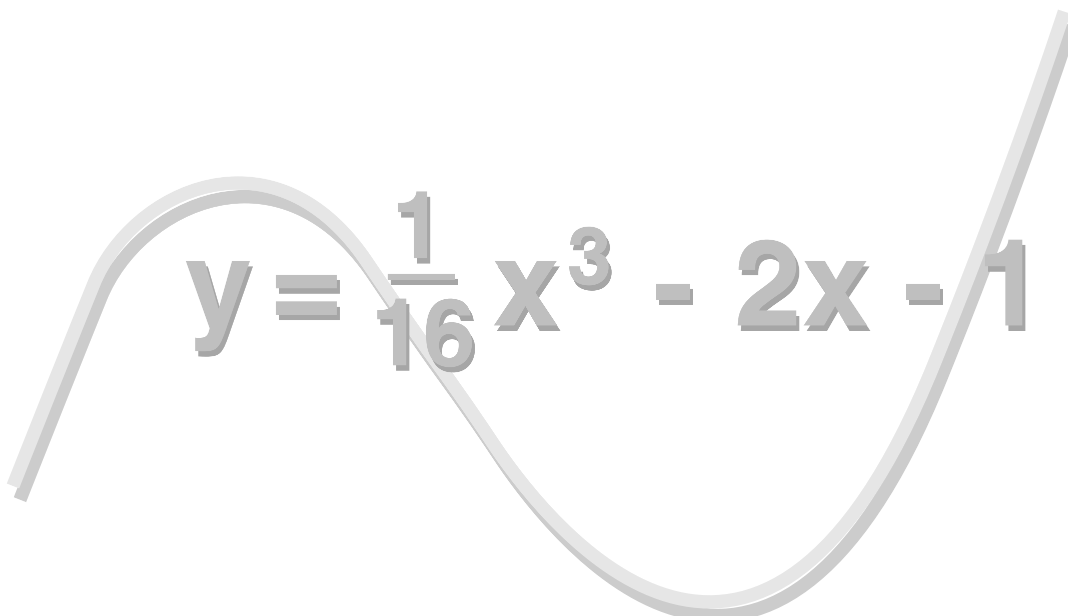
LESSON 14.1 – NONLINEAR FUNCTIONS



Here's what you'll learn in this lesson:

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Polynomial Functions

a. Recognizing polynomial functions

b. Symmetry

c. Even and odd fuctions

d. Graphing polynomial functions

e. Translations and reflections

Rational Functions

a. Recognizing rational functions

b. Determining vertical and horizontal asymptotes

c. Graphing rational functions

OVERVIEW

Polynomial functions and rational functions play a very important role in almost all branches of mathematics. They also arise in numerous applications in many other disciplines including physics, biology, and economics.

In this lesson, you will learn how to recognize and graph polynomial and rational functions.

EXPLAIN

POLYNOMIAL FUNCTIONS

Summary

Polynomial Functions

You have previously learned about many types of functions, and you have learned how to graph them. Now you are going to work with polynomial functions, and learn techniques that will help you graph them.

Here are some examples of polynomial functions:

$P(x) = 2x - 13$ $\qquad$ $P(x) = 3x^5 + 17x^2 - 5x + 1$ $\qquad$ $P(x) = x$

In general, a polynomial function is a function that can be written in this form:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$.

This is the definition of a polynomial function in one variable, x.

Here, $a_n \neq 0$ and n is a whole number.

The coefficients, a_n, a_{n-1}, a_{n-2}, and so on, can be real or nonreal complex numbers. But in this lesson you will be working with polynomials that have real coefficients only.

The coefficient a_n is called the leading coefficient. It is the coefficient of the term with the largest exponent of x. The degree of the polynomial is n.

The term a_0 is called the constant term.

Here is the first example from above rewritten in the general form of a polynomial function:

$$\begin{aligned} P(x) &= 2x - 13 \\ &= 2x + (-13) \end{aligned}$$

The leading coefficient a_1 is 2, and the degree of the polynomial function is 1.

The constant term a_0 is –13.

Similarly, here's the second example from above:

$P(x) = 3x^5 + 17x^2 - 5x + 1$

Here, $a_5 = 3$, $a_4 = 0$, $a_3 = 0$, $a_2 = 17$, $a_1 = -5$, and $a_0 = 1$.

The leading coefficient a_5 is 3, and the degree is 5.

Some simple functions are polynomial functions. For example:

$P(x) = x$ is a polynomial function of degree 1. It can be written as $P(x) = 1 \cdot x$.

$P(x) = 5$ is a polynomial function of degree 0.

Here's another polynomial function.

$$P(x) = x^2 + 17 - 5x^4 + 21x$$

If you rewrite it with the terms in descending order, (largest exponent first), it's easier to identify the coefficients and the degree:

$$P(x) = -5x^4 + x^2 + 21x + 17$$

So $a_4 = -5$, $a_3 = 0$, $a_2 = 1$, $a_1 = 21$, and $a_0 = 17$.

The leading coefficient a_4 is -5, and the degree is 4.

Graphs of Polynomial Functions

The degree of a polynomial can help you predict the shape of its graph. Here are several cases.

Polynomial Functions of Degree Zero

A polynomial function of degree zero is a function of the form $P(x) = a_0$. Its only term is the constant a_0.

Its graph is the horizontal line $y = a_0$.

Here's an example.

The polynomial $P(x) = 5$ is a polynomial function of degree zero. Its graph is the horizontal line $y = 5$. See the graph in Figure 14.1.1.

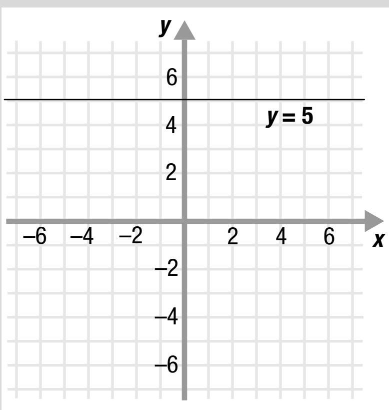


Figure 14.1.1

Polynomial Functions of Degree One

A polynomial function of degree one has an x term and a constant term, and is of the form $P(x) = a_1x + a_0$.

It is a linear equation whose graph is a line with slope a_1 and y-intercept $(0, a_0)$.

Here's an example.

The polynomial $P(x) = 4x - 7$ is a polynomial function of degree one. Its graph, the line $y = 4x - 7$, is the line with slope 4 and y-intercept $(0, -7)$. See the graph in Figure 14.1.2.

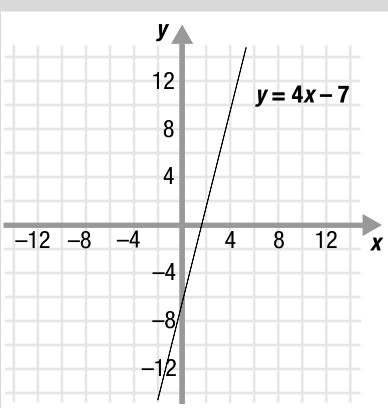


Figure 14.1.2

Polynomial Functions of Degree Two

A polynomial function of degree two has three terms, and is a function of the form $P(x) = a_2x^2 + a_1x + a_0$. Its graph is a parabola.

Here's an example.

The polynomial $P(x) = 3x^2 - 4$ is a polynomial function of degree two with $a_2 = 3$, $a_1 = 0$, and $a_0 = -4$. The graph is the parabola $y = 3x^2 - 4$. See the graph in Figure 14.1.3.

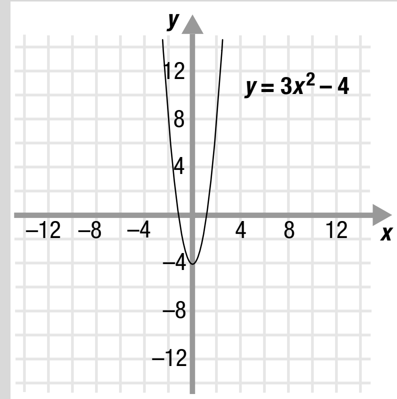


Figure 14.1.3

Symmetry

Now you'll learn some techniques that will help you graph some polynomial functions of higher degree. The first such technique is symmetry.

Symmetry About the *y*-axis

Look at the labeled points on the graph of $y = x^2 - 2$ in Figure 14.1.4.

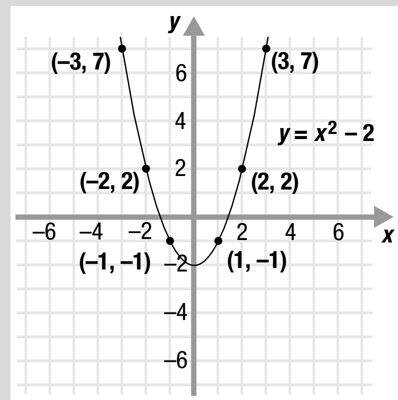


Figure 14.1.4

x	y
1	–1
–1	–1
2	2
–2	2
3	7
–3	7

Notice that these points show up in pairs, with x and $-x$ giving the same y value. For example $x = 3$ and $x = -3$ both give $y = 7$. Such pairs of points (x, y) and $(-x, y)$ are symmetric about the y-axis.

A graph is symmetric about the y-axis if when you replace x with $-x$ in the equation represented by the graph you get back the same equation.

You have symmetry about the y-axis when the y-axis acts as a mirror.

In general, to check if a graph is symmetric about the y-axis:

1. Replace x with $-x$ in the equation represented by the graph.
2. Simplify. If the equation you get is the same as the original equation, then the graph is symmetric about the y-axis.

For example, to see if the graph of $y = P(x) = x^2 - 2$ is symmetric about the y-axis:

1. Replace x with $-x$ in the equation $y = x^2 - 2$. $\qquad y = (-x)^2 - 2$
2. Simplify and check to see if you end up with the same equation. $\qquad y = x^2 - 2$

The equation you end up with is the same as the original one, and so, the graph of $y = x^2 - 2$ is symmetric about the y-axis.

Knowing that the graph of a polynomial is symmetric about the y-axis will help you graph the polynomial. If you know the graph for positive values of x, you automatically know the graph for the corresponding negative values of x. Visually, the left and right halves of the graph are mirror images of each other.

The term $5x^6$ is of degree 6.

The term $-8x^4$ is of degree 4.

The term $3x^2$ is of degree 2.

The term $7 = 7x^0$ is of degree 0.

The graph of any polynomial where all the terms have even degree is always symmetric about the y-axis.

For example, all of the terms of the polynomial $P(x) = 5x^6 - 8x^4 + 3x^2 + 7$ are of even degree, so its graph is symmetric about the y-axis.

The graph of the polynomial function $y = P(x) = 2x^3 + 8$ is not symmetric about the y-axis because $2x^3$ has odd degree. You can check this as follows:

1. Replace x with $-x$ in the equation $y = 2x^3 + 8$. $\quad y = 2(-x)^3 + 8$
2. Simplify and check to see if you end up with the same equation. $\quad y = -2x^3 + 8$

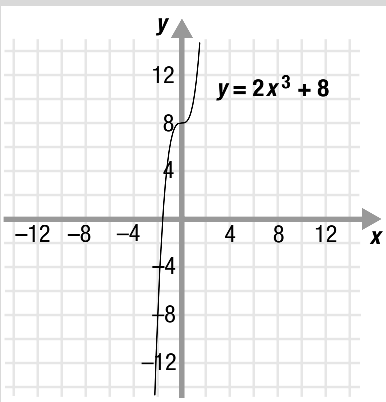


Figure 14.1.5

You do not end up with the original equation, so the graph of $y = P(x) = 2x^3 + 8$ is not symmetric about the y-axis. You can see this from the graph in Figure 14.1.5.

More generally, any function $y = f(x)$ is called an even function if $f(x) = f(-x)$ for all x in the domain of f.

Symmetry About the x-axis

You can define symmetry about the x-axis in a way corresponding to symmetry about the y-axis, by checking for pairs of points (x, y) and $(x, -y)$. Visually, if a graph is symmetric about the x-axis, its top and bottom halves are mirror images of each other.

Notice that $x = y^2 + 1$ is not the graph of a function because it fails the vertical line test. But you can still check for symmetry.

A graph is symmetric about the x-axis when you can replace y with $-y$ in the equation represented by the graph and get back the same equation.

To check if a graph is symmetric about the x-axis:

1. Replace y with $-y$ in the equation represented by the graph.
2. Simplify. If the equation you get is the same as the original equation, then the graph is symmetric about the x-axis.

For example, to see if the graph of $x = y^2 + 1$ is symmetric about the x-axis:

1. Replace y with $-y$ in the equation. $\quad x = (-y)^2 + 1$
2. Simplify and check to see if you end up with the same equation. $\quad x = y^2 + 1$

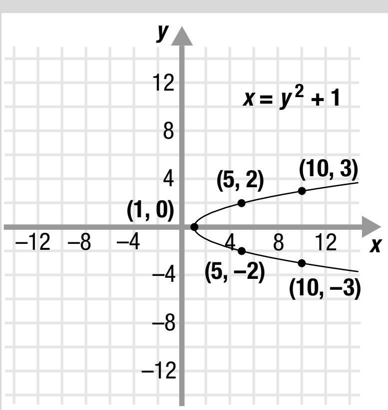


Figure 14.1.6

The equation you end up with is the same as the original one, so the graph of $x = y^2 + 1$ is symmetric about the x-axis. You can see this from the graph in Figure 14.1.6.

About the Origin

A third type of symmetry is called symmetry about the origin. Here you look at pairs of points (x, y) and $(-x, -y)$. A graph is symmetric about the origin if when you replace x with $-x$ and y with $-y$ in the equation represented by the graph, you get back the same equation.

To check if a graph is symmetric about the origin:

1. Replace x with $-x$ and replace y with $-y$ in the equation represented by the graph.
2. Simplify. If the equation you get is the same as the original equation, then the graph is symmetric about the origin.

For example, to see if the graph of $y = P(x) = 2x^3$ is symmetric about the origin:

1. Replace x with $-x$ and replace y with $-y$ in the equation $y = 2x^3$. $\quad -y = 2(-x)^3$
2. Simplify and check to see if you end up with the same equation. $\quad -y = -2x^3$, $\quad y = 2x^3$

The equation you end up with is the same as the original one, so the graph of $y = P(x) = 2x^3$ is symmetric about the origin. See the graph in Figure 14.1.7.

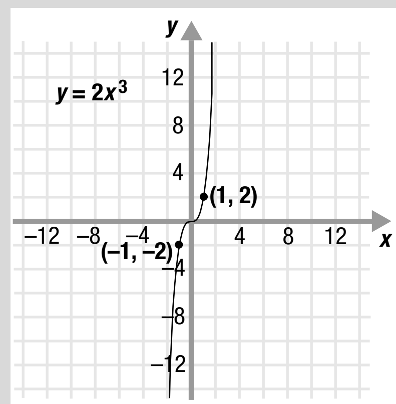


Figure 14.1.7

For a function $f(x)$, you can also check for symmetry about the origin in the following way:

1. Replace x with $-x$ in the function $f(x)$.
2. Simplify. If you get back -1 times the original function, then the graph is symmetric about the origin.

For example, to use this method to see if the graph of the function $y = f(x) = 2x^3$ is symmetric about the origin:

1. Replace x with $-x$ in the function $f(x)$. $\quad f(-x) = 2(-x)^3$
2. Simplify. If you get back -1 times the original function, then the graph is symmetric about the origin. $\quad = -2x^3$, $\quad = -1 \cdot f(x)$

The function you end up with is -1 times the original one, so the graph of $y = f(x) = 2x^3$ is symmetric about the origin.

Visually, a function is symmetric about the origin if when you rotate its graph by 180° about the origin you get back the same graph.

The graph of any polynomial where all the terms have odd degree is always symmetric about the origin.

The term $5x^7$ is of degree 7.

The term $-3x^3$ is of degree 3.

The term $8x$ is of degree 1.

For example, all of the terms of the polynomial $P(x) = 5x^7 - 3x^3 + 8x$ are of odd degree, so its graph is symmetric about the origin.

You can also see this by replacing x with $-x$ in the function:

$$\begin{aligned} P(-x) &= 5(-x)^7 - 3(-x)^3 + 8(-x) \\ &= -5x^7 + 3x^3 - 8x \\ &= -1 \cdot (5x^7 - 3x^3 + 8x) \\ &= -1 \cdot P(x) \end{aligned}$$

More generally, any function $y = f(x)$ is called an odd function if $f(-x) = -1 \cdot f(x)$ for all x in the domain of f.

Techniques for Graphing Polynomial Functions

You have previously learned how to graph a polynomial function of degree zero, one or two by creating a table of ordered pairs that satisfy the function. You'll get the graph of a horizontal line, non-horizontal line, or parabola, respectively.

When you a graph polynomial function of higher degree, it is useful to have some techniques other than a table of ordered pairs. What you just learned about symmetries is one such technique, as well as being able to factor the polynomial into a product of linear factors. Here's a method you can use to graph a polynomial function:

1. Write the function as a product of linear factors.
2. Set each factor equal to zero to find the x-intercepts.
3. Use these points to divide the x-axis into intervals.
4. In each interval, choose a test point and determine whether the function is positive or negative.
5. Check for any symmetries.
6. Use all of the information above to graph the function.

Here's an example.

To graph the polynomial function $y = P(x) = 16 - x^2$:

1. Write the function as a product of linear factors.

   $y = (4 + x)(4 - x)$

2. Set each factor equal to zero to find the x-intercepts.

   $4 + x = 0$ or $4 - x = 0$

   $x = -4$ or $x = 4$

   The x-intercepts are $(-4, 0)$ and $(4, 0)$.

3. Use these points to divide the x-axis into intervals.

The three intervals are:

$x < -4$

$-4 < x < 4$

$x > 4$

4. In each interval, choose a test point and determine whether the function is positive or negative.

Interval	$x < -4$	$-4 < x < 4$	$x > 4$
Test Point	-5	0	5
Value of $P(x)$	$16 - (-5)^2 = 16 - 25$ $= -9$	$16 - (0)^2 = 16$	$16 - (5)^2 = 16 - 25$ $= -9$
Sign of $P(x)$	$-$	$+$	$-$

5. Check for any symmetries.

$P(x)$ is symmetric about the y-axis, because $P(-x) = P(x)$.

$$P(-x) = 16 - (-x)^2 = 16 - x^2 = P(x)$$

$P(x)$ is not symmetric about the x-axis or the origin.

6. Use all of the information above to graph the function.

The function is graphed in Figure 14.1.8.

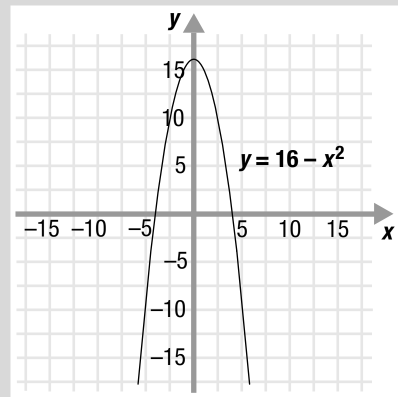


Figure 14.1.8

Here's another example.

To graph the polynomial function $y = P(x) = x^4 - x^2$:

1. Write the function as a product of linear factors.

$$P(x) = x^2(x^2 - 1) = x^2(x + 1)(x - 1)$$

2. Set each factor equal to zero to find the x-intercepts.

$x + 1 = 0 \quad x^2 = 0 \quad \text{or} \quad x - 1 = 0$

$x = -1 \quad x = 0 \quad \text{or} \quad x = 1$

The x-intercepts are $(-1, 0)$, $(0, 0)$, and $(1, 0)$.

3. Use these points to divide the x-axis into intervals.

The four intervals are:

$x < -1$

$-1 < x < 0$

$0 < x < 1$

$x > 1$

4. In each interval, choose a test point and determine whether the function is positive or negative.

Interval	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
Test Point	-2	-0.5	0.5	2
Value of $P(x)$	$(-2)^4 - (-2)^2$ $= 16 - 4$ $= 12$	$(-0.5)^4 - (-0.5)^2$ $= 0.0625 - 0.25$ $= -0.1875$	$(0.5)^4 - (0.5)^2$ $= 0.0625 - 0.25$ $= -0.1875$	$(2)^4 - (2)^2$ $= 16 - 4$ $= 12$
Sign of $P(x)$	+	−	−	+

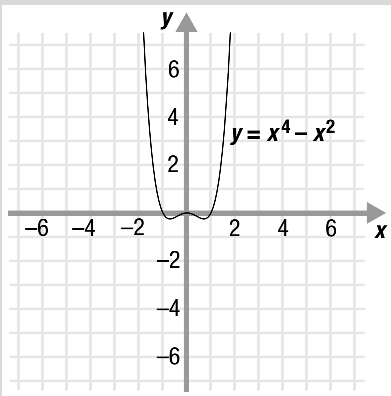


Figure 14.1.9

5. Check for any symmetries.

$P(x)$ is symmetric about the y-axis because $P(-x) = P(x)$.

$$P(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = P(x)$$

$P(x)$ is not symmetric about the x-axis or the origin.

6. Use all of the information above to graph the function.

The function is graphed in Figure 14.1.9.

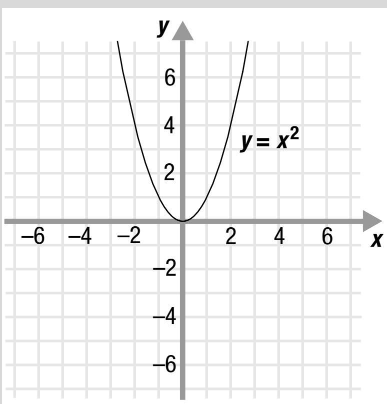


Figure 14.1.10

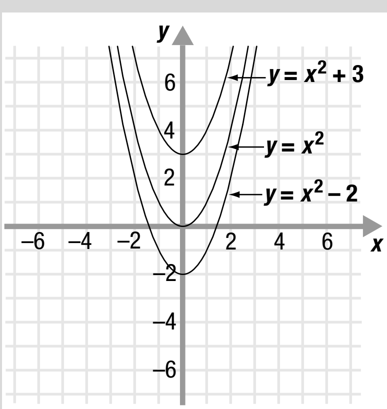


Figure 14.1.11

Translating Graphs

Once you know the graph of a particular function you can use it to help graph some closely related functions: for example, the same function that has been moved upwards, downwards, to the left or to the right.

Vertical Translations

Look at the graph of $y = x^2$ in Figure 14.1.10.

Now look at the table of points for $y = x^2$ and two other functions: $y = x^2 - 2$ and $y = x^2 + 3$.

x	$y = x^2$	$y = x^2 - 2$	$y = x^2 + 3$
0	0	−2	3
1	1	−1	4
−1	1	−1	4
2	4	2	7
−2	4	2	7
3	9	7	12
−3	9	7	12

The graphs of all of these polynomial functions are shown in Figure 14.1.11

As you can see, the three graphs have the same shape. The graph of $y = x^2 - 2$ looks like the graph of $y = x^2$, except that it is shifted down 2 units.

The graph of $y = x^2 + 3$ looks like the graph of $y = x^2$, except that it is shifted up 3 units.

In each case, the new graphs are shifts up or down the y-axis of the graph of $y = x^2$. These shifts are called vertical translations of the graph.

In general, the graph of the polynomial function $y = P(x) + k$ is the graph of $y = P(x)$ translated up k units, and the graph of $y = P(x) - k$ is the graph of $y = P(x)$ translated down k units. Here, k is a positive constant.

For example, the graph of $y = x^4 - 5x^3 + 11$ is a vertical translation up 11 units of the graph of $y = x^4 - 5x^3$.

Horizontal Translations

You can also translate graphs by moving them horizontally to the left or to the right. Here is a table of points for three functions: $y = x^2$, $y = (x - 2)^2$ and $y = (x + 1)^2$.

x	$y = x^2$	$y = (x - 2)^2$	$y = (x + 1)^2$
0	0	4	1
1	1	1	4
–1	1	9	0
2	4	0	9
–2	4	16	1
3	9	1	16
–3	9	25	4

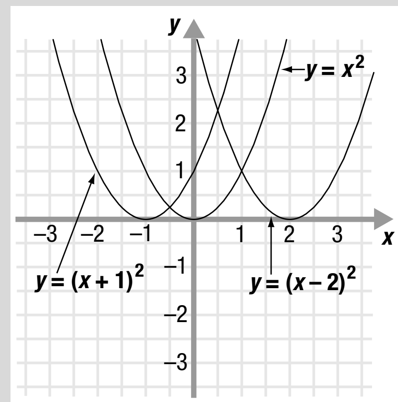


Figure 14.1.12

The graphs of these three polynomial functions are shown in Figure 14.1.12.

As you can see, all three graphs have the same shape. The graph of $y = (x - 2)^2$ looks like the graph of $y = x^2$, except that it is shifted right 2 units. The graph of $y = (x + 1)^2$ looks like the graph of $y = x^2$, except that it is shifted left 1 unit.

These shifts are called horizontal translations of the graph.

In general, the graph of the polynomial function $y = P(x - k)$ is the graph of $y = P(x)$ translated right k units, and the graph of $y = P(x + k)$ is the graph of $y = P(x)$ translated left k units. Here, k is a positive constant.

For example, the graph of $y = (x + 7)^5$ is a horizontal translation 7 units to the left of the graph of $y = x^5$.

Be careful:

*y = P(x **– k**) translates the graph of y = P(x) to the **right.***

*y = P(x **+ k)** translates the graph of y = P(x) to the **left.***

Graphing $y = ax^n$

There are other changes that you can make to the graph of $y = x^2$ to help you graph closely related functions. Look at the following table of points for the functions $y = x^2$, $y = -x^2$, $y = 3x^2$, $y = \frac{1}{2}x^2$:

x	$y = x^2$	$y = -x^2$	$y = 3x^2$	$y = \frac{1}{2}x^2$
0	0	0	0	0
1	1	–1	3	0.5
–1	1	–1	3	0.5
2	4	–4	12	2
–2	4	–4	12	2
3	9	–9	27	4.5
–3	9	–9	27	4.5

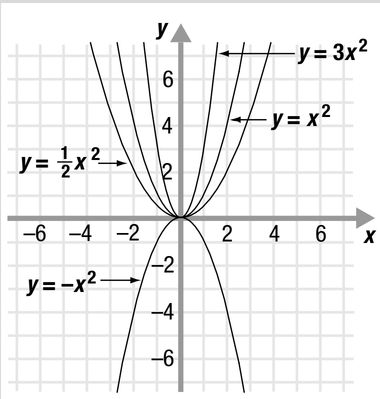


Figure 14.13

The graphs of these functions are shown in Figure 14.1.13.

Notice that if you put a negative sign in front of the x^2 term, you turn the graph of $y = x^2$ upside down. So the graph of $y = -x^2$ is the graph of $y = x^2$, reflected about the x-axis.

If you multiply the x^2 term of $y = x^2$ by 3, you get the function $y = 3x^2$, whose graph is narrower and steeper than the graph of $y = x^2$.

If you multiply the x^2 term of $y = x^2$ by $\frac{1}{2}$, you get the function $y = \frac{1}{2}x^2$, whose graph is wider and less steep than the graph of $y = x^2$.

In general, for the parabola $y = ax^2$:

- If a is positive, the parabola opens upwards.
- If a is negative, the parabola opens downwards.
- If $|a| > 1$ the parabola is narrower than $y = x^2$.
- If $|a| < 1$ the parabola is wider than $y = x^2$.

Sample Problems

Answers to Sample Problems

1. Find the leading coefficient and the degree of the polynomial function $P(x) = -13 + x^4 - 8x^7 + 5x^2$.

   ✔ a. Arrange the terms in descending order. $P(x) = -8x^7 + x^4 + 5x^2 - 13$

   ☐ b. Find the leading coefficient. __________

   ☐ c. Find the degree of the polynomial. __________

b. – 8

c. 7

2. Match each polynomial function in the left column with a graph shape in the right column.

Functions	Shapes
$P(x) = 5x^2 - 17$	Horizontal line
$Q(x) = -14x + 3$	Parabola
$R(x) = 23$	Non-horizontal line

☑ a. The graph of $P(x) = 5x^2 - 17$ is a: Parabola

☐ b. The graph of $Q(x) = -14x + 3$ is a: ____________

☐ c. The graph of $R(x) = 23$ is a: ____________

3. Is the graph of the equation $y = 5x^7 - 3x^3 + 11x$
   a. symmetric about the y-axis?
   b. symmetric about the x-axis?
   c. symmetric about the origin?

☑ a. Test for symmetry about the y-axis.

- Replace x with $-x$ in the original equation, $y = 5x^7 - 3x^3 + 11x$. — $y = 5(-x)^7 - 3(-x)^3 + 11(-x)$
- Simplify. — $= -5x^7 + 3x^3 - 11x$
- Do you end up with the same equation? Is there symmetry about the y-axis? — You don't end up with the original equation, so the graph is not symmetric about the y-axis.

☐ b. Test for symmetry about the x-axis.

- Replace y with $-y$ in the original equation, $y = 5x^7 - 3x^3 + 11x$. ____________
- Do you end up with the same equation? Is there symmetry about the x-axis? ____________

☐ c. Test for symmetry about the origin.

- Replace y with $-y$ and x with $-x$ in the original equation, $y = 5x^7 - 3x^3 + 11x$. ____________
- Simplify. ____________
- Do you end up with the same equation? Is there symmetry about the origin? ____________

Answers to Sample Problems

b. Non-horizontal line

c. Horizontal line

b. $-y = 5x^7 - 3x^3 + 11x$

You don't end up with the original equation, so the graph is not symmetric about the x-axis.

c. $-y = 5(-x)^7 - 3(-x)^3 + 11(-x)$

$y = 5x^7 - 3x^3 + 11x$

You do end up with the original equation, so the graph is symmetric about the origin.

Answers to Sample Problems

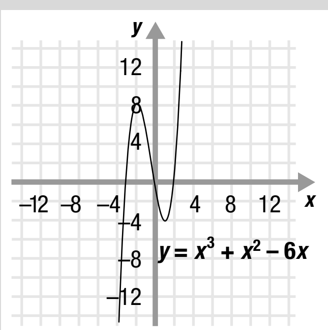
b. *−3, 0, 2*

c. *$x < -3$*
$-3 < x < 0$
$0 < x < 2$
$x > 2$

d. *negative*
positive
negative
positive

e. *No*
No
No

f.



4. Graph the polynomial function $y = P(x) = x^3 + x^2 - 6x$.

☑ a. Write the function as a product of linear factors.

$$\begin{aligned} P(x) &= x^3 + x^2 - 6x \\ &= x(x^2 + x - 6) \\ &= x(x - 2)(x + 3) \end{aligned}$$

☐ b. Set each factor equal to zero to find the x-intercepts.

The x-intercepts are (_____, 0) (_____, 0), and (_____, 0),

☐ c. Use these points to divide the x-axis into intervals. List these intervals.

____
____
____
____

☐ d. In each interval, choose a test point to determine the sign of the function.

Interval	Sign of $P(x)$
$x < -3$	_________
$-3 < x < 0$	_________
$0 < x < 2$	_________
$x > 2$	_________

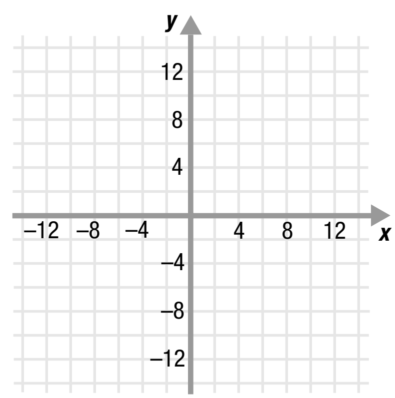
☐ e. Check for any symmetries.

Is the graph symmetric about the x-axis? _________

y-axis? _________

origin? _________

☐ f. Use all of the information above to graph the function.

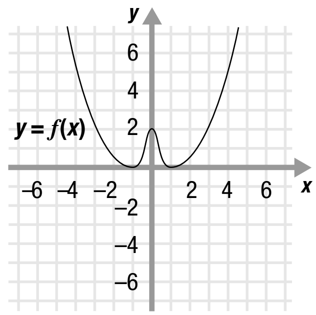


5. Here is the graph of $y = f(x)$.

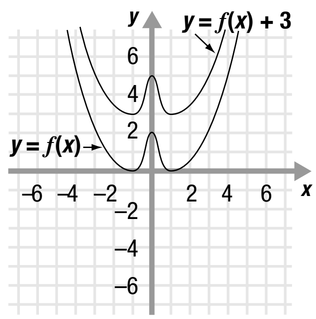
a. Graph $y = f(x) + 3$.

b. Graph $y = f(x + 1)$.

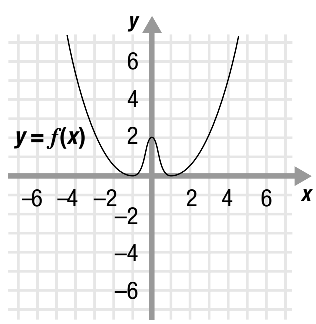
c. Graph $y = f(x + 1) + 3$.



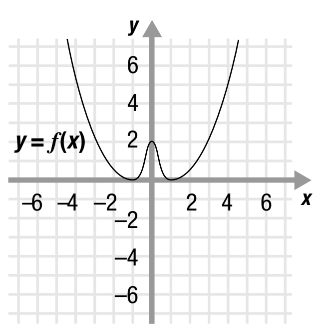
☑ a. Graph $y = f(x) + 3$.
Translate the original graph up 3 units.



☐ b. Graph $y = f(x + 1)$.

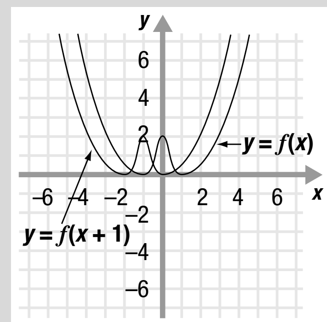


☐ c. Graph $y = f(x + 1) + 3$.

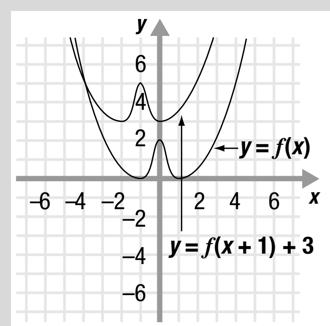


Answers to Sample Problems

b. Translate the original graph 1 unit to the left.



c. Translate the original graph 3 units up and 1 unit to the left.



RATIONAL FUNCTIONS

Summary

Rational Functions

A rational function is a quotient of two polynomial functions. It can be written in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions. Since division by zero is not defined, any value of x which makes $Q(x) = 0$ is excluded from the domain of a rational function.

Here are some examples of rational functions:

$f(x) = \frac{2}{x}$ $\quad g(x) = \frac{3x^5 + 17x^2 - 5x + 1}{x^2 - 1}$ $\quad h(x) = \frac{2x - 1}{x(x+1)}$

$k(x) = 2 + \frac{3}{x} - \frac{4}{x^3}$

Even though $k(x) = 2 + \frac{3}{x} - \frac{4}{x^3}$ is not of the form $\frac{P(x)}{Q(x)}$ it can be written in this form: $\frac{2x^3 + 3x^2 - 4}{x^3}$

The Graph of a Rational Function

The graph of a rational function looks quite different from the graph of a polynomial function. One reason for this is the behavior of the graph near x values that make the denominator zero.

Here is a table of ordered pairs for $f(x) = \frac{2}{x}$.

x	$f(x) = \frac{2}{x}$
−4	−0.5
−2	−1
−1	−2
−0.5	−4
0.5	4
1	2
2	1
4	0.5

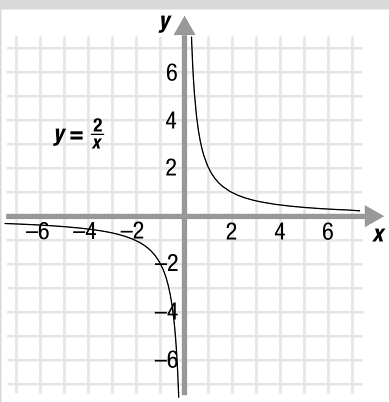


Figure 14.1.14

Look at the graph of $y = f(x) = \frac{2}{x}$ in Figure 14.1.14.

The value $x = 0$ is not in the domain of f because it makes the denominator zero.

Notice the following for $y = f(x) = \frac{2}{x}$:

- There is a break in the graph at $x = 0$.
- For values of x near zero (positive or negative), the graph gets close to, but never touches the vertical line $x = 0$.
- For large values of x (positive or negative) the graph gets close to the horizontal line $y = 0$.

Here are some more examples.

The graph of the rational function $y = g(x) = \frac{(x+1)}{x(x-2)}$ is shown in Figure 14.1.15.

Notice the breaks in the graph at $x = 0$ and at $x = 2$. These are the values of x that make the denominator zero. Near $x = 0$ the graph gets close to a vertical line (the y-axis), and near $x = 2$ the graph gets close to a vertical line ($x = 2$).

In addition, for positive and negative large values of x, the graph gets close to a horizontal line (the x-axis).

Now look at the graph of $y = h(x) = \frac{10}{(x-2)(x+3)}$, shown in Figure 14.1.16.

Notice the breaks in the graph at $x = -3$ and at $x = 2$. These are the values of x that make the denominator zero. Near $x = -3$ the graph gets close to a vertical line ($x = -3$), and near $x = 2$ the graph gets close to a vertical line ($x = 2$).

In addition, for positive and negative large values of x, the graph gets close to a horizontal line (the x-axis).

To graph polynomial functions such as these you can create a table of ordered pairs, but there are some other techniques that may help.

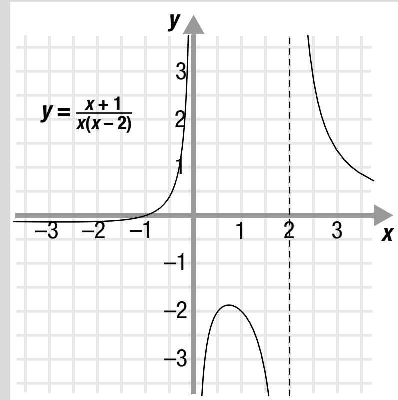


Figure 14.1.15

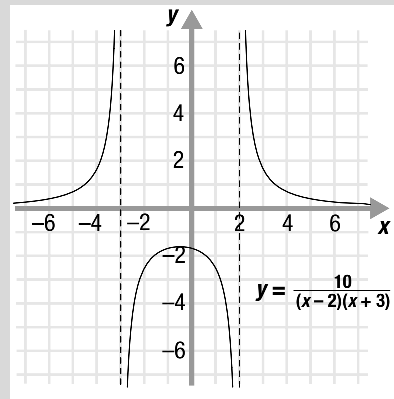


Figure 14.1.16

Vertical Asymptotes

At the values of x where the denominator of a rational function is zero, the rational function is not defined, and there is a break in the graph.

To examine this more closely, look again at the graph of $y = f(x) = \frac{2}{x}$, shown in Figure 14.1.17.

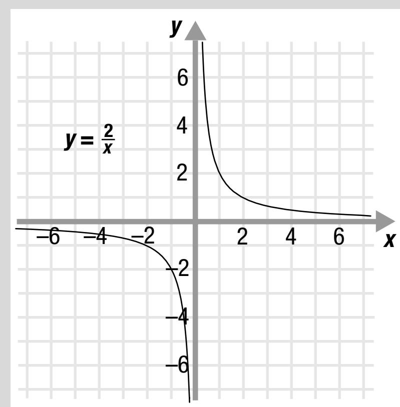


Figure 14.1.17

Here is a table of ordered pairs for values of x that approach zero from the left:

x	$f(x) = \frac{2}{x}$
-4	-0.5
-2	-1
-1	-2
-0.5	-4
-0.1	-20
-0.05	-40
-0.01	-200
-0.005	-400

As you can see, as x gets closer and closer to 0 from the left, $f(x)$ becomes a larger and larger negative number.

You can write: As $x \to 0$ from the left, $f(x) \to -\infty$.

Read "$\to$" as "approaches".

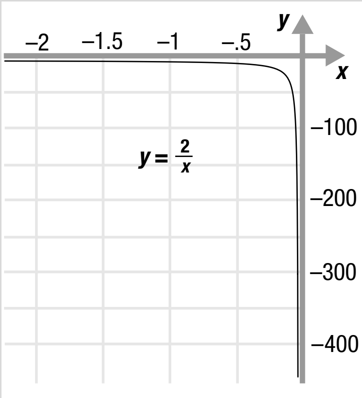


Figure 14.1.18

The view of the graph as x approaches 0 from the left is shown enlarged in Figure 14.1.18.

Here is a table of ordered pairs for values of x that approach zero from the right:

x	$f(x) = \frac{2}{x}$
0.005	400
0.01	200
0.05	40
0.1	20
0.5	4
1	2
2	1
4	0.5

As x gets closer and closer to 0 from the right, $f(x)$ becomes a larger and larger positive number.

You can write: As $x \to 0$ from the right, $f(x) \to \infty$.

The vertical line $x = 0$ that the graph approaches is called a vertical asymptote of the rational function $f(x) = \frac{2}{x}$.

A vertical asymptote for the function $f(x)$ is a vertical line with the property that the graph of $f(x)$ gets very close to this line, but never intersects it.

When a rational function is reduced to lowest terms, the graph of the rational function has a vertical asymptote for each value of x that makes the denominator of the rational function zero.

In general, to find any vertical asymptote for the rational function $f(x) = \frac{P(x)}{Q(x)}$:

1. Reduce the rational function $\frac{P(x)}{Q(x)}$ to lowest terms.
2. Find the values of x that make the denominator zero.
3. If $x = a$ is a value you found in step 2, then the vertical line $x = a$ is a vertical asymptote.

For example, to find the vertical asymptotes for a rational function you've already seen, $y = h(x) = \frac{10}{(x+3)(x-2)}$:

1. The rational function is already reduced to lowest terms.	
2. Find the values of x that make the denominator zero.	$x = -3$ and $x = 2$.
3. If $x = a$ is a value you found in step 2, then $x = a$ is a vertical asymptote.	The vertical line $x = -3$ is a vertical asymptote, and the vertical line $x = 2$ is a vertical asymptote.

The graph of $y = h(x) = \frac{10}{(x+3)(x-2)}$ is shown in Figure 14.1.19. The vertical asymptotes are shown as dotted lines.

You can write: As $x \to -3$ from the left, $f(x) \to \infty$.

As $x \to -3$ from the right, $f(x) \to -\infty$.

As $x \to 2$ from the left, $f(x) \to -\infty$.

As $x \to 2$ from the right, $f(x) \to \infty$.

Horizontal Asymptotes

The graph of a rational function may approach a horizontal line for large positive or negative values of x. A horizontal line with the property that the graph of $f(x)$ gets very close to this line as $x \to \infty$ or as $x \to -\infty$, but never reaches, is called a horizontal asymptote.

Look again at the graph of $y = f(x) = \frac{2}{x}$, shown in Figure 14.1.20.

Here's a table of some points on the graph:

x	$f(x) = \frac{2}{x}$
0.5	4
1	2
2	1
4	0.5
8	0.125
10	0.1
50	0.02
100	0.01

A closer view of the graph as x gets large positively is shown in Figure 14.1.21.

On this graph, as the values of x get larger and larger positively, the values of $y = f(x)$ get closer and closer to zero.

You can write: As $x \to \infty$, $f(x) \to 0$.

Similarly, as the values of x get larger and larger negatively, the values of $y = f(x)$ get closer and closer to zero. You can write: As $x \to -\infty$, $f(x) \to 0$.

You can combine the two statements and write: As $|x| \to \infty$, $f(x) \to 0$.

In general, to find the horizontal asymptote for the function $f(x) = \frac{P(x)}{Q(x)}$:

1. Make sure that the rational function $\frac{P(x)}{Q(x)}$ is reduced to lowest terms and that the degree of $P(x)$ is less than or equal to the degree of $Q(x)$.

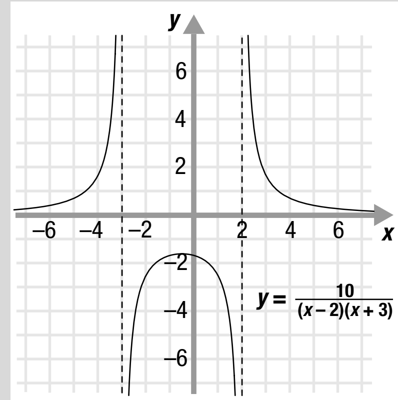


Figure 14.1.19

A rational function may touch or cross a horizontal asymptote when x is not large. However, when x gets very large, it will not touch or cross a horizontal asymptote.

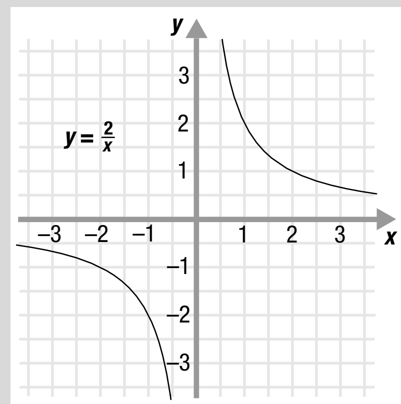


Figure 14.1.20

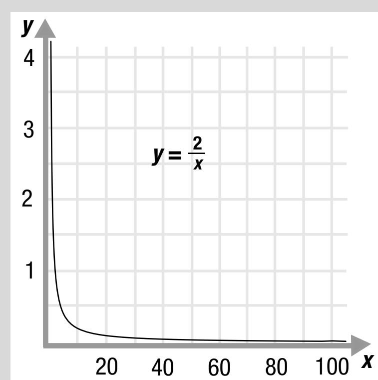


Figure 14.1.21

2. Change the form of the rational function by dividing the numerator and denominator by the highest power of x present in the rational function.

3. Look at what happens to $f(x)$ as $|x| \to \infty$. The parts containing x will approach zero and $f(x)$ will approach a constant b.

4. The horizontal line $y = b$ is a horizontal asymptote.

Here's an example.

To find the horizontal asymptote for the rational function $y = f(x) = \frac{2x}{x-3}$:

1. The rational function is already reduced to lowest terms and the degree of $2x$ is equal to the degree of $x - 3$.

2. Change the form of the rational function by dividing the numerator and denominator by x, the highest power of x present in the rational function.

$$= \frac{\frac{2x}{x}}{\frac{x-3}{x}}$$

$$= \frac{2}{1 - \frac{3}{x}}$$

3. Look at what happens to $f(x)$ as $|x| \to \infty$. The parts containing x will approach zero and $f(x)$ will approach a constant b.

As $|x| \to \infty$, $\frac{3}{x} \to 0$ and $f(x) \to 2$.

4. The horizontal line $y = b$ is a horizontal asymptote.

The line $y = 2$ is a horizontal asymptote.

The graph is shown in Figure 14.1.22.

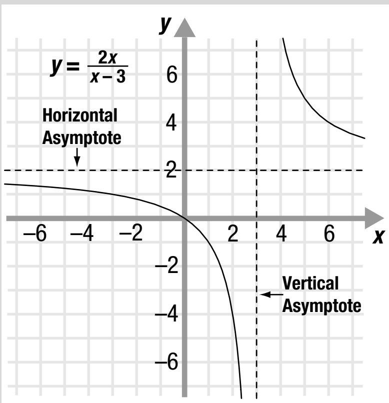


Figure 14.1.22

You may have noticed that the graph of a rational function never touches or crosses a vertical asymptote. But it may touch or cross a horizontal asymptote when x is not large. However, when x gets very large, it will not touch or cross a horizontal asymptote.

Oblique Asymptotes

When a rational function has a numerator whose degree is one greater than that of the denominator its graph does not approach a horizontal asymptote. Instead, as $|x| \to \infty$, the graph approaches an oblique (or slant) line. Here's an example.

Look at the graph of $y = f(x) = \frac{2x^2 + x + 1}{x}$, shown in Figure 14.1.23.

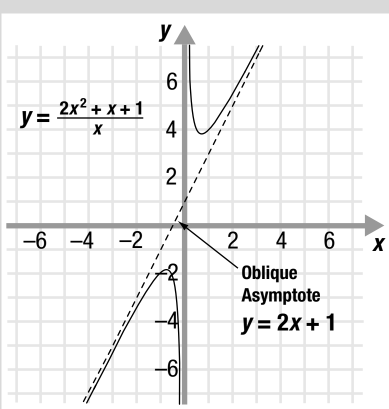


Figure 14.1.23

The line $y = 2x + 1$ is an oblique asymptote. You can write:

As $|x| \to$, $f(x) \to 2x + 1$.

Here are the general steps to find an oblique asymptote:

1. Make sure that the rational function $\frac{P(x)}{Q(x)}$ is reduced to lowest terms and that the degree of $P(x)$ is one more than the degree of $Q(x)$.

2. Use long division to divide $Q(x)$ into $P(x)$. You'll get $ax + b + \frac{c}{Q(x)}$, where a, b, and c are constants.

3. As $|x| \to \infty$, the remainder $\frac{c}{Q(x)} \to 0$, and $f(x) \to ax + b$.

4. The line $y = ax + b$ is an oblique asymptote for the rational function $\frac{P(x)}{Q(x)}$.

For example, to find an oblique asymptote for the rational function $f(x) = \frac{2x^2 + x + 1}{x}$:

1. The rational function is already reduced to lowest terms and the degree of $2x^2 + x + 1$ is one more than the degree of x.

2. Use long division to put $\frac{2x^2 + x + 1}{x}$ in the form $ax + b + \frac{c}{Q(x)}$, where a, b, c are constants.

   $2x + 1 + \frac{1}{x}$

3. As $|x| \to \infty$ the remainder $\frac{c}{Q(x)} \to 0$, and $f(x) \to ax + b$.

   As $|x| \to \infty$, $\frac{1}{x} \to 0$, and $f(x) \to 2x + 1$.

4. The line $y = ax + b$ is an oblique asymptote.

   The line $y = 2x + 1$ is an oblique asymptote.

Notice that a rational function can have several vertical asymptotes. But it can have only one horizontal asymptote or one oblique asymptote; it cannot have both.

Graphing Rational Functions

If you combine what you've learned about asymptotes, along with the steps used for graphing a polynomial function, you can graph a rational function.

Here are the steps you'll use to graph the rational function $y = \frac{P(x)}{Q(x)}$.

1. Reduce the rational function to lowest terms.
2. Find all vertical asymptotes.
3. Find a horizontal or an oblique asymptote.
4. Find the x- and y- intercepts.
5. Use any vertical asymptotes and x-intercepts to divide the x-axis into intervals.
6. Choose test points to determine the sign of $\frac{P(x)}{Q(x)}$ in each interval.
7. Test for symmetries.
8. Graph the function.

Here's an example.

To graph $y = f(x) = \frac{2x}{6x - 4}$:

1. Reduce the rational function to lowest terms.

$$f(x) = \frac{2x}{2(3x-2)} = \frac{x}{3x-2}$$

2. Find all vertical asymptotes.
   - These values of x that make the denominator zero.

$3x - 2 = 0$

The line $x = \frac{2}{3}$ is a vertical asmptote.

3. Find a horizontal or oblique asymptote.
   - The degree of the numerator equals the degree of the denominator, so there is a horizontal asymptote. To find this asymptote, divide the numerator and denominator by x, let $|x| \to \infty$ and see what happens to $f(x)$.

$f(x) = \dfrac{1}{3 - \frac{2}{x}}$

As $|x| \to \infty$, $\frac{2}{x} \to 0$,

and $f(x) \to \frac{1}{3}$.

The line $y = \frac{1}{3}$ is a horizontal asymptote.

4. Find x- and y- intercepts.
   - To find the y-intercept, set $x = 0$ and solve for y.

     The y-intercept is (0, 0).
   - To find the x-intercept, set $y = 0$ and solve for x.

     The x-intercept is (0, 0).

5. Use the x-intercept and the vertical asymptote to divide the x-axis into intervals.

The intervals are: $x < 0$

$0 < x < \frac{2}{3}$

$x > \frac{2}{3}$

6. Choose a test point in each interval to determine the sign of $f(x)$.

Interval	$x < 0$	$0 < x < \frac{2}{3}$	$x > \frac{2}{3}$
Test Point	–1	$\frac{1}{3}$	1
Value of $P(x)$	$\dfrac{-1}{3(-1) - 2} = \dfrac{1}{5}$	$\dfrac{\frac{1}{3}}{3\left(\frac{1}{3}\right) - 2} = \dfrac{\frac{1}{3}}{-1} = -\dfrac{1}{3}$	$\dfrac{1}{3(1) - 2} = 1$
Sign of $f(x)$	+	−	+

7. Test for symmetries.

$$f(-x) = \frac{-x}{3(-x) - 2} = \frac{x}{3x + 2} \neq f(x) \text{ or } -f(x)$$

The graph of $f(x)$ is not symmetric about the y-axis, the x-axis, or the origin.

8. Graph the function.

The graph of $y = f(x) = \dfrac{2x}{6x - 4}$ is shown in Figure 14.1.24.

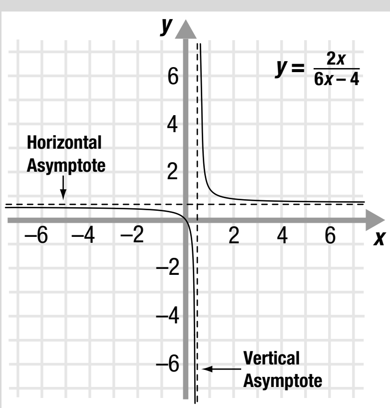


Figure 14.1.24

To help you graph a function you may also wish to plot a few points that lie on the graph.

Sample Problems

1. Find the domain of the rational function below and identify any x-values where there is a break in its graph:

   $f(x) = \frac{3x + 4}{6x^2 - x - 2}$

   ✔ a. Factor the denominator. $6x^2 - x - 2$ $= (3x - 2)(2x + 1)$

   ☐ b. Set each factor equal to zero and solve for x. $x =$ ________ $x =$ ________

   ☐ c. Write the domain.

   ☐ d. Identify x-values where there is a break in the graph. $x =$ ________ $x =$ ________

2. Find the vertical and horizontal asymptotes for the rational function $f(x) = \frac{3x^2 - 4}{x(x + 2)}$.

   ✔ a. Check that the rational function is reduced to lowest terms. The function is reduced to lowest terms.

   ☐ b. Find the values of x that make the denominator zero. ________________

   ☐ c. Write the vertical asymptotes. ________________

   ✔ d. Check that the degree of $3x^2 - 4$ is less than or equal to the degree of $x(x + 2)$. The degree of $3x^2 - 4$ is 2. The degree of $x(x + 2)$ is 2.

   ☐ e. Change the form of the rational function by dividing the numerator and the denominator by x^2. $f(x) =$ ________

   ☐ f. Look at what happens to $f(x)$ as $|x| \to \infty$. As $|x| \to \infty$, $f(x) \to$ ________

   ☐ g. Write the horizontal asymptote. ________

Answers to Sample Problems

b. $\frac{2}{3}, -\frac{1}{2}$

c. All real numbers except $x = \frac{2}{3}$ *and* $x = -\frac{1}{2}$

d. $\frac{2}{3}, -\frac{1}{2}$

b. $x = 0$ *or* $x = -2$

c. $x = 0$ *and* $x = -2$

e. $\dfrac{3 - \frac{4}{x^2}}{1 + \frac{2}{x}}$

f. 3

g. $y = 3$

Answers to Sample Problems

b. $x = -2$

c. $x = -2$

e. $x + 1 - \frac{2}{x + 2}$

f. $x + 1$

g. $y = x + 1$

b. $-2, 3$

c. $x = -2$ and $x = 3$

3. Find the vertical and oblique asymptotes for the rational function $f(x) = \frac{x^2 + 3x}{x + 2}$.

☑	a.	Check that the rational function is reduced to lowest terms.	The function is reduced to lowest terms.
☐	b.	Find the value(s) of x that make the denominator zero.	________
☐	c.	Write the vertical asymptote.	________
☑	d.	Check that the degree of $x^2 + 3x$ is greater than the degree of $x + 2$.	The degree of $x^2 + 3x$ is 2. The degree of $x + 2$ is 1.
☐	e.	Use long division to put $\frac{x^2 + 3x}{x + 2}$ in the form $ax + b + \frac{c}{Q(x)}$, where a, b, c are constants.	$f(x) =$ ________
☐	f.	Look at what happens to $f(x)$ as $\lvert x \rvert \to \infty$.	As $\lvert x \rvert \to \infty$, $f(x) \to$ ________
☐	g.	Write the oblique asymptote.	________

4. Graph the rational function $y = f(x) = \frac{3x^2}{x^2 - x - 6}$.

☑	a.	Reduce the rational function to lowest terms.	The function $f(x)$ is already reduced to lowest terms.
☐	b.	Find any vertical asymptotes. • Find the x-values that make the denominator zero.	$x =$ ________ or $x =$ ________
☐	c.	• Write the vertical asymptotes.	________________

☑ d. Find a horizontal or an oblique asymptote.
- Compare the degree of the numerator and the denominator.

They have the same degree, so there is a horizontal asymptote.

☐ e.
- Divide the numerator and the denominator by x^2. ____________
- Look at what happens to $f(x)$ as $|x| \to \infty$.

As $|x| \to \infty$, $f(x) =$ ________

☐ f.
- Write the horizontal asymptote. ____________

☑ g. Find the x- and y-intercepts. Here there is only one intercept.

(0, 0)

☐ h. Use the x-intercept and the vertical asymptotes to divide the x-axis into intervals. List these intervals.

____
____
____
____

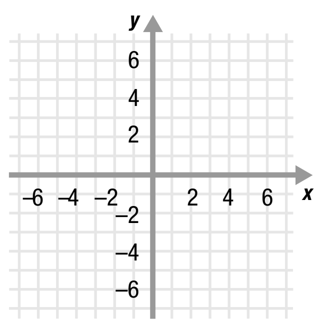
☐ i. In each interval, choose a test point to determine the sign of the function.

Interval	Sign of $f(x)$
$x < -2$	________
$-2 < x < 0$	________
$0 < x < 3$	________
$x > 3$	________

☐ j. Check for any symmetries.

Is the graph symmetric about the
x-axis? ________
y-axis? ________
origin? ________

☐ k. Use all of the information above to graph the function.



Answers to Sample Problems

e. $\dfrac{3}{1 - \frac{1}{x} - \frac{6}{x^2}}$

3

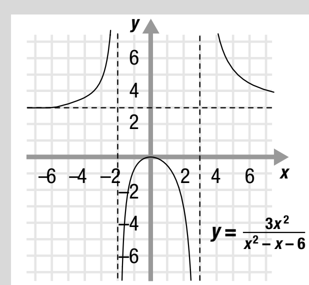
f. $y = 3$

h. $x < -2$
$-2 < x < 0$
$0 < x < 3$
$x > 3$

i. *positive*
negative
negative
positive

j. *No*
No
No

k.



Answers to Sample Problems

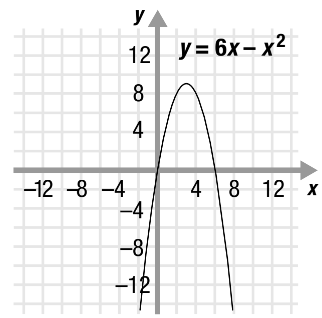
EXPLORE

Sample Problems

On the computer you explored how a function changes when you translate its graph up or down, or to the left or right. Below are some additional exploration problems.

1. Below is the graph of the polynomial function $y = P(x) = 6x - x^2$. Translate the graph of $P(x)$ to graph the polynomial function $y = Q(x) = 3 + 6x - x^2$.

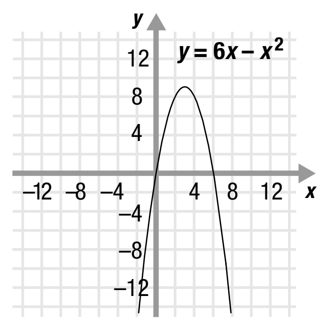
   ☑ a. Here is the graph of the polynomial function $y = P(x) = 6x - x^2$.



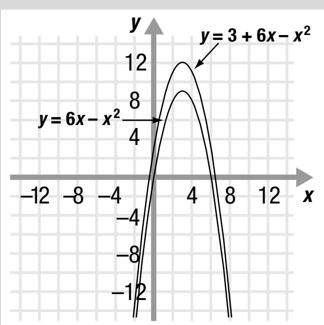
   ☐ b. The graph of $y = Q(x) = 3 + 6x - x^2$ is ___ units up/down from the graph of $y = P(x) = 6x - x^2$.

b. 3 units up

   ☐ c. Graph $y = Q(x)$.

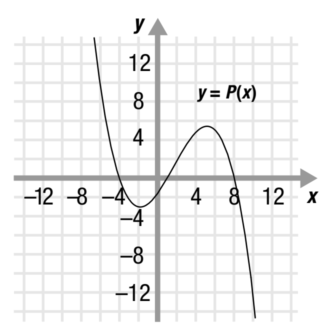


c.



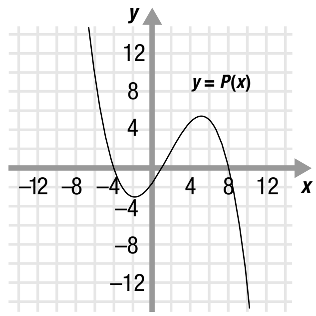
2. Below is the graph of the function $y = P(x)$. Translate the graph of $y = P(x)$ to graph the polynomial function $y = P(x + 4)$.

   ☑ a. Here is the graph of the function $y = P(x)$.



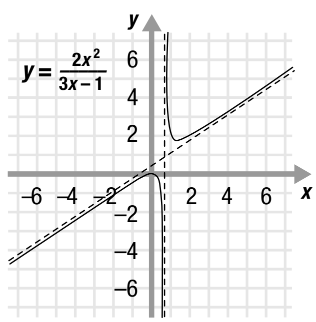
☐ b. The graph of $y = P(x + 4)$ is ___units to the left/right of the graph of $y = P(x)$.

☐ c. Graph $y = P(x + 4)$.



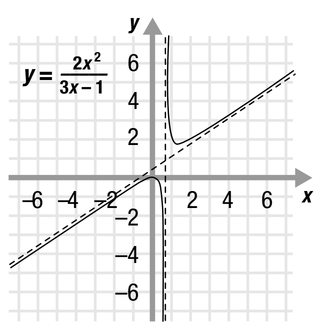
3. Below is the graph of the rational function $y = f(x) = \frac{2x^2}{3x-1}$. Translate the graph of $f(x)$ to graph the rational function $y = g(x) = \frac{2x^2}{3x-1} - 4$.

☑ a. Here is the graph of the rational function $y = f(x) = \frac{2x^2}{3x-1}$.



☐ b. The graph of $y = g(x) = \frac{2x^2}{3x-1} - 4$ is ___ units up/down from the graph of $y = f(x) = \frac{2x^2}{3x-1}$.

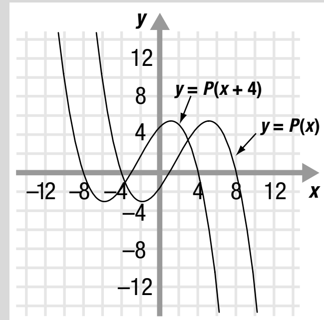
☐ c. Graph $y = g(x)$.



Answers to Sample Problems

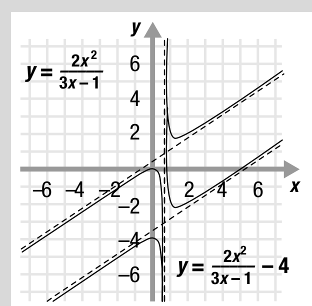
b. 4 units to the left

c.



b. 4 units down

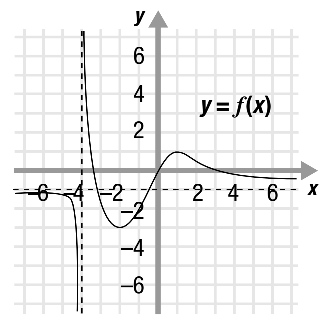
c.



Answers to Sample Problems

4. Below is the graph of the function $y = f(x)$. Translate the graph of $y = f(x)$ to graph the polynomial function $y = f(x-2) + 5$.

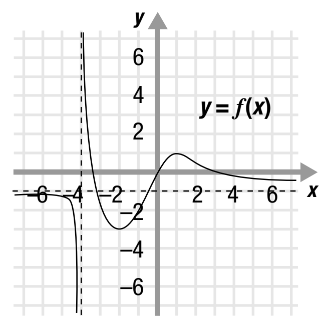
☑ a. Here is the graph of the function $y = f(x)$.



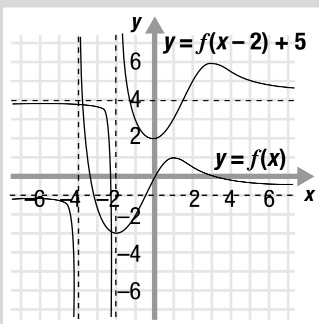
☐ b. The graph of $y = f(x-2) + 5$ is ___ units to the left/right of and ___ units up/down from the graph of $y = f(x)$.

b. 2 units to the right of and 5 units up

☐ c. Graph $y = f(x-2) + 5$.



c.



HOMEWORK

Homework Problems

Circle the homework problems assigned to you by the computer, then complete them below.

Explain

Polynomial Functions

1. Identify the leading coefficient and the degree of the polynomial function $15x^9 - 3x^2 - 8$.

2. Is the graph of $y^4 + y^2 - 7 = 3x + 5x^3$ symmetric about the x-axis?

3. Identify the graph of the polynomial function $y = P(x) = -4x^2$ in Figure 14.1.25.

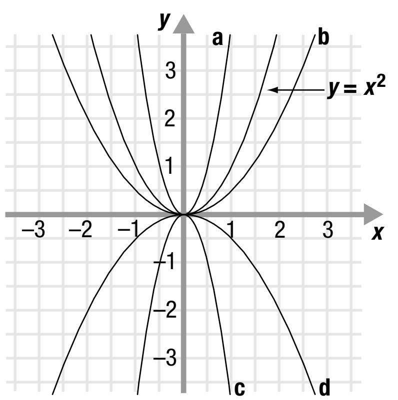


Figure 14.1.25

4. Identify the leading coefficient, the constant term, and the degree of the polynomial function $13x^3 - 15x^4 - 3x^2 - 27x^5 + 23$.

5. Is the polynomial function $P(x) = 3x^2 + 2x$ symmetric about the origin?

6. Identify the graph of the polynomial function $y = P(x) = (x - 2)^3$ in Figure 14.1.26.

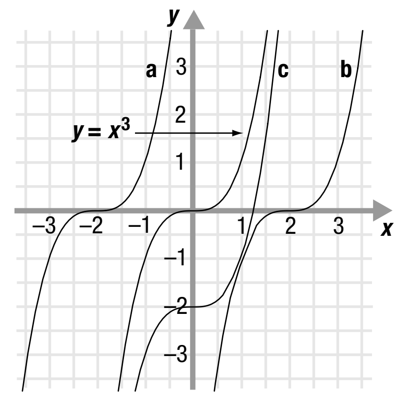


Figure 14.1.26

7. Which of the polynomial functions below could be the one graphed in Figure 14.1.27?

   $y = P(x) = a_1x + a_0$

   $y = P(x) = a_2x^2 + a_1x + a_0$

   $y = P(x) = a_3x^3 + a_1x + a_0$

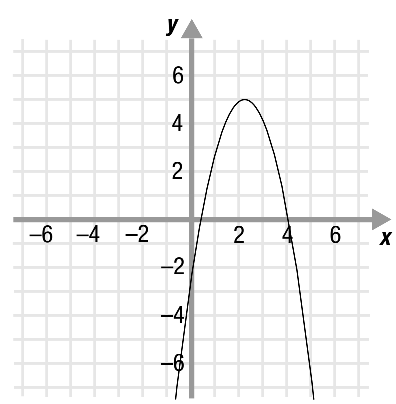


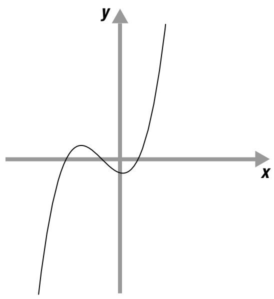
Figure 14.1.27

8. Graph the polynomial function $y = P(x) = x^3 - x$.

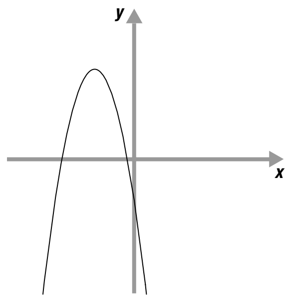
9. Graph the following.
   a. $y = x^2$
   b. $y = x^2 - 3$
   c. $y = (x + 1)^2$

10. Which graph below could be the correct graph of the polynomial function $y = P(x) = a_3x^3 + a_2x^2 + a_0$, where a_3 is a negative number?

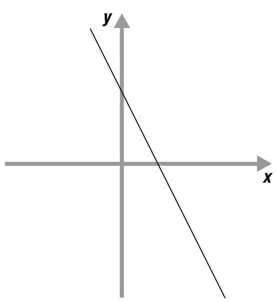
a.



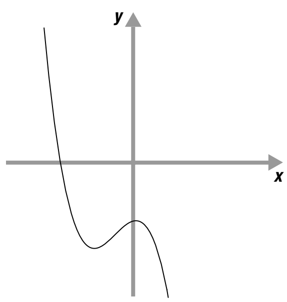
b.



c.



d.



11. Graph the polynomial function $y = P(x) = x^4 - x^2$.

12. Consider the polynomial function $Q(x) = 3(x - 2)^2 - 5$. Choose the correct statement below.
    Compared to the graph of $P(x) = x^2$, the graph of $Q(x)$ is:
    a. The same shape, 2 units to the right, and 5 units down.
    b. A narrower shape, 2 units to the left, and 5 units up.
    c. A wider shape, 2 units to the right, and 5 units down.
    d. A narrower shape, 2 units to the right, and 5 units down.
    e. A wider shape, 2 units to the left, and 5 units down.

Rational Functions

13. Find the domain of the rational function $f(x) = \frac{28x^3}{7 + x}$. Identify any x-values where there is a break in the graph.

14. Find all of the asymptotes of the graph of the rational function $f(x) = \frac{3}{x^2}$.

15. Graph the rational function $y = f(x) = \frac{1}{x^2}$.

16. Find the domain of the rational function $f(x) = \frac{x(x + 1)}{15x^2 + x - 2}$. Identify any x- values where there is a break in its graph.

17. Find all of the asymptotes of the graph of the rational function $f(x) = \frac{4x^2 + 7}{3(x + 1)(x - 2)}$

18. Graph the rational function $y = f(x) = \frac{3x}{x^2 - 5x + 6}$.

19. Find the vertical asymptotes for the graph of the rational function $f(x) = \frac{2x - 3}{x^2 - x - 12}$.

20. Find all the asymptotes for the graph of the rational function $f(x) = \frac{3x^2 - 2}{x + 3}$.

21. Graph the rational function $f(x) = \frac{3 - 4x}{x + 1}$.

22. Find the horizontal asymptote for the graph of the rational function $f(x) = \frac{5x^2 + 3x}{x^2 - x - 12}$.

23. Find all of the asymptotes for the graph of the rational function $f(x) = \frac{4x^3 + 5x^2 + 6}{x^2 - 3x - 4}$.

24. Graph the rational function $y = f(x) = \frac{2x^3 + 5x + 7}{x(x - 3)}$.
    Hint: The only x-intercept is (–1, 0).

Explore

25. Suppose you are given the graph of a function $y = f(x)$. If you translate this graph down 3 units and to the right 6 units, you get a new graph. Write the function described by this graph.

26. The parabola $y = 3x^2$ is graphed in Figure 14.1.28. Graph the parabola $y = 3x^2 - 5$ on the same grid.

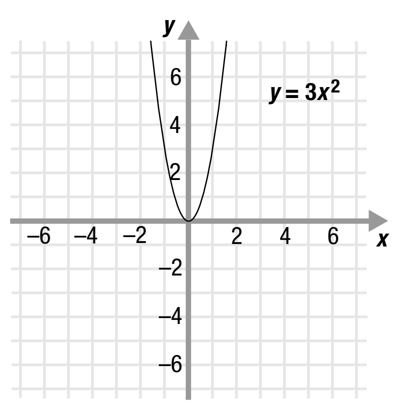


Figure 14.1.28

27. The function $y = f(x)$ is graphed in Figure 14.1.29. Graph the function $y = f(x + 2)$ on the same grid. Write the equations of the new asymptotes.

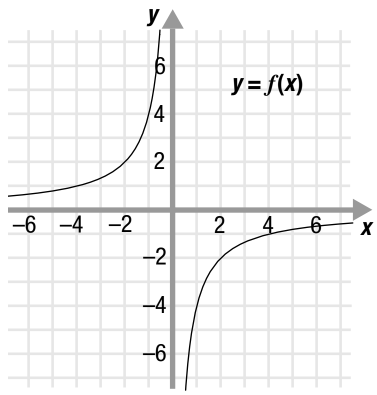


Figure 14.1.29

28. Suppose you are given the graph of the rational function $f(x) = \frac{x^2}{x-2}$. If you translate this graph up 2 units and to the left 7 units, you get a new graph. Write the function g described by this graph.

29. The rational function $y = f(x) = \frac{1}{(x-2)(x+1)}$ is graphed in Figure 14.1.30. Graph the rational function $y = g(x) = \frac{1}{x(x+3)}$ on the same grid. Write the vertical asymptotes for $f(x)$. Write the vertical asymptotes for $g(x)$.

Hint: g is a translation of f to the left.

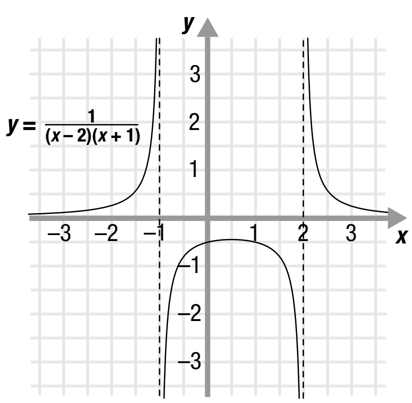


Figure 14.1.30

30. A function $y = f(x)$ is graphed in Figure 14.1.31. Graph the function $y = f(x - 1) + 6$ on the same grid. What is the equation of the new horizontal asymptote?

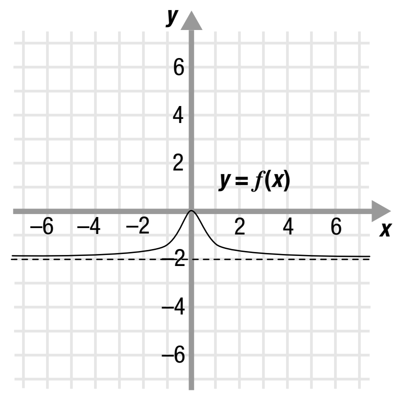


Figure 14.1.31

APPLY

PRACTICE PROBLEMS

Here are some additional practice problems for you to try.

Polynomial Functions

1. Find the leading coefficient, the constant term, and the degree of the polynomial function

   $P(x) = 13x^3 - 7x^5 + 12x^4 - 8.$

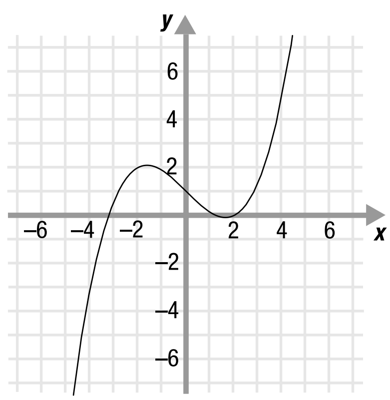
2. Determine if the polynomial function $P(x) = x^2 + 5x - 8$ is symmetric about the x-axis, the y-axis, the origin, or none of these.

3. Determine if the polynomial function $P(x) = x^3 - 5x^7$ is symmetric about the x-axis, the y-axis, the origin, or none of these.

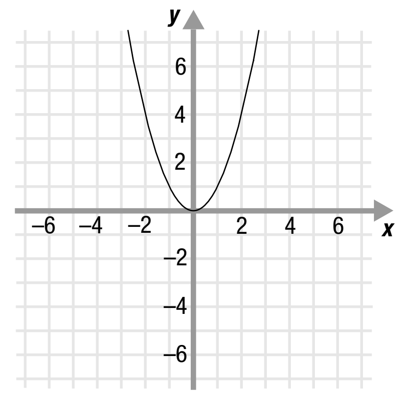
4. Graph the polynomial function $y = P(x) = x^3 - 3x^2$.

5. Graph the polynomial function $y = P(x) = x^4 + 2x^3$.

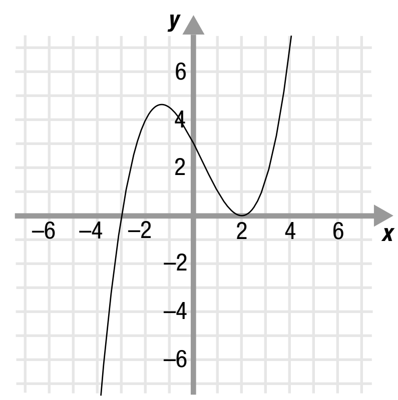
6. Here is the graph of a function $y = f(x)$. Graph $y = -f(x)$ on the same grid.



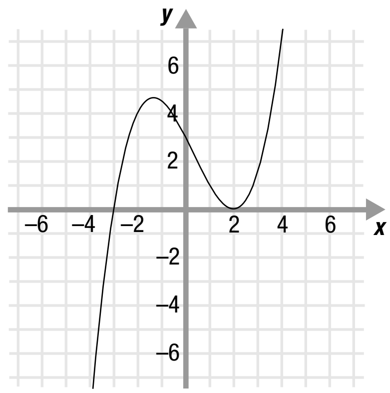
7. Here is the graph of $y = 2x^2$. Graph $y = 3x^2$ and $y = 0.5x^2$ on the same grid.



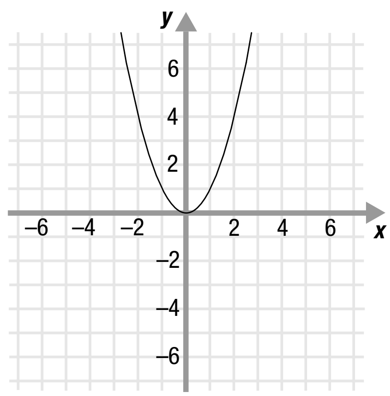
8. Here is the graph of a function $y = f(x)$. Graph $y = f(x) + 2$ on the same grid.



9. Here is the graph of a function $y = f(x)$. Graph $y = f(x - 4)$ on the same grid.



10. Here is the graph of $y = x^2$. Graph $y = (x + 2)^2 + 1$ on the same grid.



Rational Functions

11. Find the domain of the rational function $f(x) = \frac{7x(2x - 5)}{24x^2 + 14x - 3}$.

    Identify any x values where there is a break in its graph.

12. Find any vertical asymptotes for the rational function $y = f(x) = \frac{7 + x}{2x^2 + 11x - 6}$.

13. Find any horizontal asymptotes for the rational function $y = f(x) = \frac{-3x}{x - 14}$.

14. Find any oblique asymptotes for the rational function $y = f(x) = \frac{2x^3 + x}{x^2 - 1}$.

15. Find all the asymptotes for the rational function $y = f(x) = \frac{2x^2 + x - 5}{x^2 - 5x + 6}$.

16. Graph the rational functions $y = f(x) = \frac{5}{x}$ and $y = g(x) = \frac{5}{x + 2}$.

17. Graph the rational function $y = f(x) = \frac{5x^2 + 1}{x^2 + 3x - 4}$.

18. Graph the rational function $y = f(x) = \frac{x^3 + x^2}{x^2 - 9}$.

19. Find all the asymptotes for the rational function $y = f(x) = \frac{2x^3 + x}{(x - 3)(x + 1)}$.

20. Graph the rational functions $y = f(x) = \frac{5}{x}$ and $y = g(x) = \frac{5}{x} - 2$.

EVALUATE

PRACTICE TEST

Take this practice test to be sure that you are prepared for the final quiz in Evaluate.

1. Find the constant term, leading coefficient and degree of the polynomial function $P(x) = 17 - 5x^2 + 13x^5 - 14x^7 + 8x^6$.

   Is the graph of the function a parabola?

2. Test the polynomial function $y = P(x) = 5x^7 - 11x^3 + 8x$ for symmetry about the x-axis, for symmetry about the y-axis, and for symmetry about the origin.

3. Graph the polynomial function $y = P(x) = 4x - x^3$.

4. A polynomial function $y = P(x)$ is graphed in Figure 14.1.32

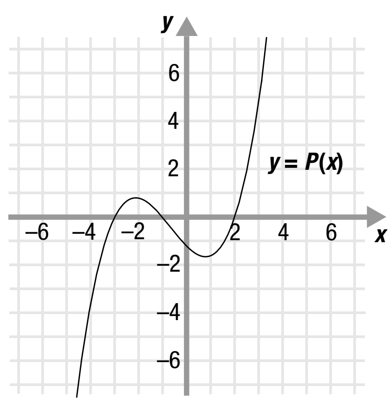


Figure 14.1.32

   a. Identify the graph of $y = P(x - 2)$ in Figure 14.1.33.

   b. Identify the graph of $y = P(x) + 2$ in Figure 14.1.33.

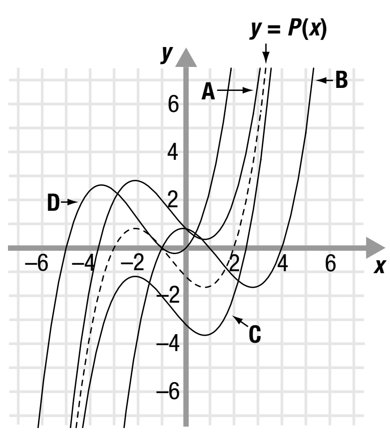


Figure 14.1.33

5. Find the domain of the rational function $f(x) = \dfrac{2x(x-3)}{2x^2+5x-12}$. Identify any x-values where there is a break in its graph.

6. Find all the asymptotes for the rational function $f(x) = \dfrac{3x}{(x+2)(x-5)}$.

7. Find all the asymptotes of the rational function $f(x) = \dfrac{5x^2-2}{x-3}$.

8. Graph the rational function $y = f(x) = \dfrac{5x^2-20}{(x-3)(x+1)}$.

9. Suppose you are given the graph of the rational function $f(x) = \dfrac{x^3}{4x^2-2}$. If you translate this graph up 8 units and to the right 3 units, you get a new graph. Write the function g described by this graph.

10. The graph of the rational function $y = f(x) = \dfrac{1}{(x-3)(x+2)}$ is graphed in Figure 14.1.34.

    Graph the rational function $y = g(x) = \dfrac{1}{(x-3)(x+2)} - 4$ on the same grid.

    Write the horizontal asymptotes for $f(x)$ and for $g(x)$.

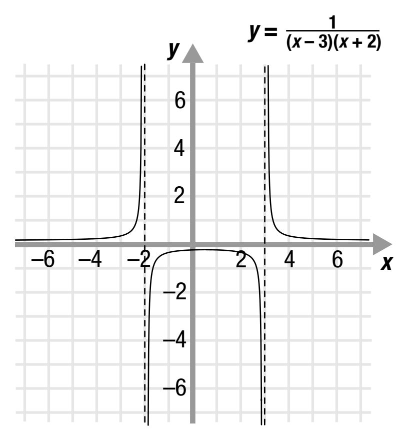


Figure 14.1.34

11. The rational function $y = f(x) = \frac{1}{(x-3)(x+2)}$ is graphed in Figure 14.1.35.

Graph the rational function $y = g(x) = \frac{1}{(x+1)(x+6)}$ on the same grid.

Write the vertical asymptotes for $f(x)$ and for $g(x)$.

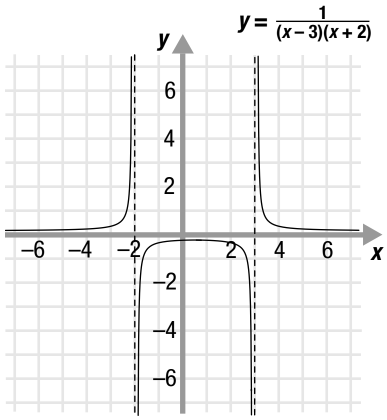


Figure 14.1.35

12. The rational function $y = f(x) = \frac{1}{x^2}$ is graphed in Figure 14.1.36.

Graph the rational function $y = g(x) = \frac{1}{(x+2)^2} - 3$ on the same grid.

Write the asymptotes for $f(x)$ and for $g(x)$.

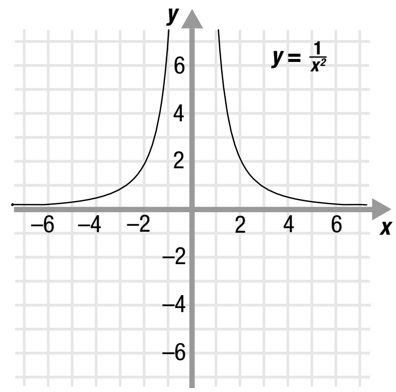


Figure 14.1.36