Test 4
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Math 125 – Spring 2009

Name: key.

It is not worth an intelligent man's time to be in the majority. By definition, there are already enough people to do that.

G.H. Hardy (1877 - 1947)
English mathematician

No work = no credit

No Symbolic Calculators

Warm-ups (1 pt each) $-3^2 =$ -9 $\int_{-1}^{1} x\,dx =$ 0 $\int_{0}^{\pi} \sin(x)\,dx =$ 2

1.) (1 pt) Paraphrase the quote by Hardy (see above). Use complete sentences.

Smart folks are in the minority.

2.) (10 pts) Determine whether the integral $\int_{3}^{\infty} \frac{dx}{x(\ln x)^2}$ converges or diverges. Evaluate the integral if it converges.

$$\int_3^\infty \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \int_3^t \frac{dx}{x(\ln x)^2}$$

$u = \ln x$

$du = \frac{dx}{x}$

$$= \lim_{t\to\infty} \int_{\ln 3}^{\ln(t)} \frac{du}{u^2}$$

$$= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_{\ln 3}^{\ln t}$$

$$= \lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 3}\right)$$

$$= \frac{1}{\ln 3} \quad \text{(converges).}$$

14

69

3.) (10 pts) Consider the definite integral $\int_0^1 \frac{4\,dx}{1+x^2}$.

a.) (2 pts) Evaluate the definite integral using the Fundamental Theorem of Calculus. Hint: the answer is π.

$$\int_0^1 \frac{4}{1+x^2}\,dx = \Big[4\arctan x\Big]_0^1$$

$$= 4\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$= \pi$$

b.) (8 pts) Evaluate $\int_0^1 \frac{4\,dx}{1+x^2}$ by using Simpson's Rule with $n=4$ to find an approximation for π (to 9 decimal places, $\pi = 3.141592654$). What is the exact error to 9 decimal places in your approximation (error = abs(exact value – approximate value))?

$$Y_1 = \frac{4}{1+x^2}$$

$$L_1 = \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1\} \;;\; \Delta x = 1/4$$

$$L_2 = y_1(L_1)$$

$$L_3 = \{1, 4, 2, 4, 1\}$$

$$L_4 = L_2 * L_3$$

$$S_4 = \frac{1/4}{3}\,\text{sum}(L_4)$$

Approximation: 3.141568627

Exact error: 2.403×10^{-5}

10

4.) (10 pts) Use the comparison test to determine whether $\int_2^\infty \frac{dx}{\sqrt[7]{x^6-1}}$ converges or diverges.

Make sure you clearly state your conclusion.

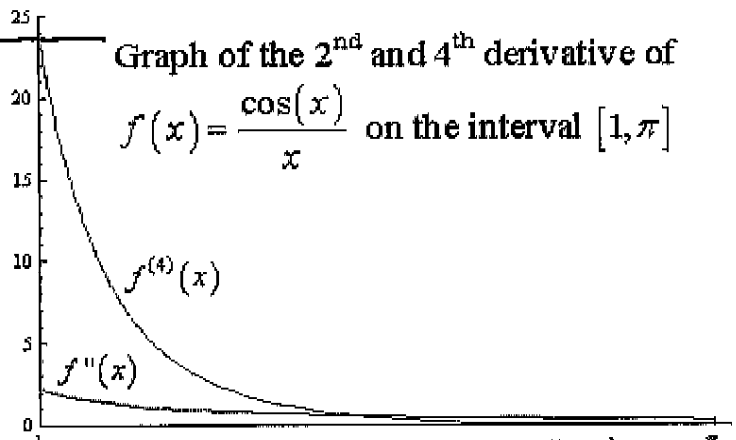
$$\int_2^\infty \frac{dx}{\sqrt[7]{x^6-1}} \geq \int_2^\infty \frac{dx}{x^{6/7}}$$

$$= \underbrace{\int_1^\infty \frac{dx}{x^{6/7}}}_{\text{divergent by the p-test.}} - \underbrace{\int_1^2 \frac{dx}{x^{6/7}}}_{\text{definite integral}}$$

Hence, the improper integral diverges by the comparison test.

5.) (10 pts) How large do we have to choose n so that the approximation S_n using Simpson's Rule to $\int_1^\pi \frac{\cos(x)}{x}dx$ has an error bound smaller than 0.00001? Use the given graph to estimate the coefficient k in the error bound.

$k = 24$



Graph of the 2^{nd} and 4^{th} derivative of $f(x) = \frac{\cos(x)}{x}$ on the interval $[1, \pi]$

$$|E_S| \leq \frac{k(b-a)^5}{180 n^4} \leq 0.00001$$

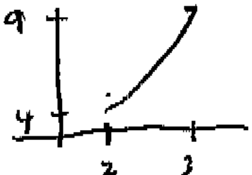
$$\frac{24(\pi-1)^5}{180 n^4} \leq 0.00001$$

(note: 27.54 if $k \doteq 23$).

$$\Rightarrow \sqrt[4]{\frac{24(\pi-1)^5}{180(0.00001)}} \leq n = 27.84$$

choose $n = 28$.

20



$x = \sqrt{y}$

$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

$\frac{dy}{dx} = 2x$

6.) (15 pts) Consider the curve $y = x^2$ on the interval $4 \le y \le 9$.

a.) <u>Set up</u> an integral representing the arc length of the curve. <u>Do not evaluate the integral</u>.

$$L = \int_2^3 \sqrt{1 + (2x)^2}\, dx$$

OR

$$L = \int_4^9 \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy \approx 5.1003$$

b.) <u>Set up</u> an integral representing the surface area of the surface formed by rotating the curve about the x-axis. <u>Do not evaluate the integral</u>.

$$SA = \int_2^3 2\pi x^2 \sqrt{1 + (2x)^2}\, dx$$

OR

$$SA = \int_4^9 2\pi y \sqrt{1 + \left(\frac{1}{2y}\right)^2}\, dy. \approx 208.42$$

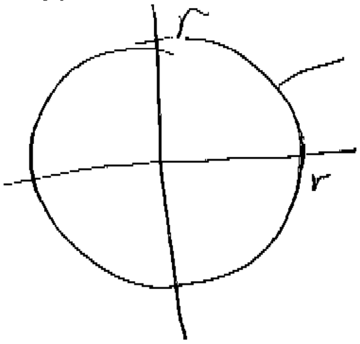
c.) <u>Set up</u> an integral representing the surface area of the surface formed by rotating the curve about the vertical line $x = -5$. <u>Do not evaluate the integral</u>.

$$SA = \int_2^3 2\pi (x + 5) \sqrt{1 + (2x)^2}\, dx$$

$$= \int_4^9 2\pi (\sqrt{y} + 5) \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy. \approx 241.373$$

$\overline{15}$

7.) (10 pts) Derive the circumference of a circle with radius r by finding the arc length of the appropriate function and using symmetry (where necessary).



$y = \sqrt{r^2 - x^2}$ on $[0, r]$

$\frac{dy}{dx} = \frac{-x}{\sqrt{r^2 - x^2}}$

$$L = 4 \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= 4r \int_0^r \frac{dx}{\sqrt{r^2 - x^2}}$$

$$= 4r \left[\arcsin\left(\frac{x}{r}\right) \right]_0^r$$

$$= 4r \left(\underbrace{\sin^{-1}(1)}_{\pi/2} - \underbrace{\sin^{-1}(0)}_{0} \right)$$

$$= 2\pi r.$$