Test 2
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Math 125 – Spring 2009

Name: key

It is unworthy of excellent men to lose hours like slaves in the labor of calculation which could safely be relegated to anyone else if machines were used.

Gottfried Wilhelm von Leibniz (1646 - 1716)
German mathematician

No work = no credit

No Symbolic Calculators

Warm-ups (1 pt each) $\frac{1}{0} =$ und $\qquad 1^0 = 1 \qquad 0^1 = 0$

1.) (1 pt) Paraphrase the quote (see above) by Leibniz in the context of this calculator and computer age. Use complete sentences.

Once we know how to solve a problem, we should turn it over to a secretary w/a calculator.

2.) (3 pts) What three real and present day countries share a land border with Mexico?

i.) Belize
ii.) Guatemala
iii.) USA

3.) (10 pts) Find the general solution to the differential equation $y' = y^4 \cos(x)$.

$$\frac{dy}{dx} = y^4 \cos x$$

$$\Rightarrow \int \frac{dy}{y^4} = \int \cos x \, dx$$

$$\Rightarrow -\frac{1}{3} y^{-3} = \sin x + C_1$$

$$\Rightarrow \frac{1}{y^3} = -3 \sin x + C_2$$

$$\Rightarrow y^3 = \frac{1}{C_2 - 3 \sin x}$$

$$y = \sqrt[3]{\frac{1}{C_2 - 3 \sin x}}$$

or $y = 0$.

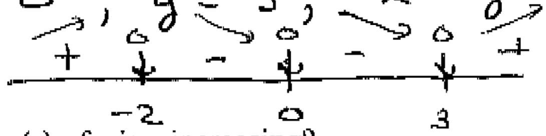
b/o for $y = \sqrt[4]{c e^{\sin x}}$

4.) (10 pts) A function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = y^4 - y^3 \mp 6y^2$.

a.) What are the constant solutions of the equation?

$\frac{dy}{dt} = 0 = y^2(y-3)(y+2)$

$\Rightarrow y = 0,\ y = 3,$ OR $y = -2$



9 pts

b.) For what value(s) of y is y increasing?

on $(-\infty, -2)$ and also on $(3, \infty)$

c.) For what value(s) of y is y decreasing?

on $(-2, 0)$ and on $(0, 3)$

d.) Classify the constant solutions from (a.) as stable or unstable solutions.

Dropped off exam,

$y = -2$: ~~un~~stable $\qquad$ $y = 3$ unstable.

$y = 0$: unstable

5.) (10 pt) Evaluate the definite integral $\int_1^6 x\sqrt{x+3}\,dx = \int_4^9 (u-3)\sqrt{u}\,du \leftarrow 5/10.$

Let $u = x + 3 \leftarrow 4/10$

$u - 3 = x$

$du = dx$

$u(1) = 4$

$u(6) = 9$

1 pt for limits

$= \int_4^9 u^{3/2} - 3u^{1/2}\,du$

$= \frac{2}{5}u^{5/2} - 2u^{3/2}\Big|_4^9$

$= \frac{2}{5}(243) - 2(27) - \left(\frac{2}{5}(32) - 2(8)\right)$

$= \frac{232}{5}$

$\frac{232}{5} = 46.4$

6.) (30 pts) Solve the differential equation $\frac{dy}{dx} = \frac{y\sin(x)}{1+y^2}$ that satisfies the initial condition $y(0) = 2$

$$\int \frac{y\,dy}{1+y^2} = \int \sin x \, dx \qquad 4/10$$

$$\Rightarrow \frac{1}{2}\ln|1+y^2| = -\cos x + c_1$$

5 pts

$$\Rightarrow \ln|1+y^2| = -2\cos x + c_2$$

$$\Rightarrow |1+y^2| = c_3 e^{-2\cos x}, \quad c_3 > 0$$

$$\Rightarrow 1+y^2 = \pm c_3 e^{-2\cos x}$$

$$\Rightarrow y^2 = -1 + c_4 e^{-2\cos x}$$

$$\Rightarrow y = \pm\sqrt{-1 + c_4 e^{-2\cos x}}$$

$$y(0) = 2$$

$$2 = \pm\sqrt{-1 + c_4 e^{-2}}$$

$$\Rightarrow 2 = \sqrt{\frac{c_4}{e^2} - 1}$$

$$\Rightarrow 4 = \frac{c_4}{e^2} - 1$$

$$\Rightarrow 5 = c_4/e^2$$

$$\Rightarrow c_4 = 5e^2 \approx 36.95$$

$$y = \sqrt{(5e^2)e^{-2\cos x} - 1}$$

7.) (10 pt) Find the exact average value of $f(x) = \sin(x)\cdot e^{\cos(x)}$ over the interval $\frac{\pi}{6} \le x \le \frac{\pi}{2}$.

$$f_{ave} = \frac{1}{\frac{\pi}{2} - \frac{\pi}{6}} \int_{\pi/6}^{\pi/2} \sin x \, e^{\cos x} \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$u(\pi/2) = 0$

$u(\pi/6) = \sqrt{3}/2$

$$= \frac{3}{\pi} \int_{\frac{\sqrt{3}}{2}}^{0} -e^{u} \, du.$$

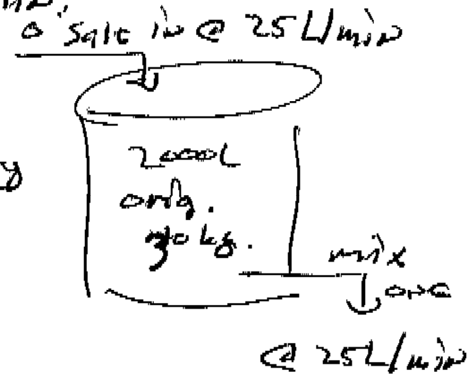
$$= -\frac{3}{\pi} \left[e^{u} \right]_{\frac{\sqrt{3}}{2}}^{0}$$

$$= -\frac{3}{\pi}\left(1 - e^{\sqrt{3}/2}\right)$$

$$\frac{3}{\pi}\left(e^{\sqrt{3}/2} - 1\right)$$

8.) (10 pts) A tank is filled with 2000 L of brine with 40kg of dissolved salt. Pure water is pumped into the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How long until only 5kg of salt remains?

$A(t)$ = amt of salt (kg) after t min.



$$\frac{dA}{dt} = (\text{rate in}) - (\text{rate out})$$

$$= 0 - \frac{A}{2000} \cdot 25 \qquad \frac{4}{10} \text{ if any DE}$$

$$\Rightarrow \frac{dA}{A} = -\frac{25}{2000}\, dt \qquad 6 \text{ pts}$$

$$\Rightarrow \ln|A| = -\frac{1}{80}t + C_1$$

$$\Rightarrow |A| = C_2 e^{-\frac{1}{80}t}$$

$$\Rightarrow A = C_3 e^{-\frac{1}{80}t}$$

$$A(0) = 40 = C_3$$

$\Rightarrow$ solve

$$5 = 40 e^{-1/80\, t}$$

$$\Rightarrow \frac{1}{8} = e^{-1/80\, t}$$

$$\Rightarrow \ln\left(\frac{1}{8}\right) = -\frac{1}{80}t$$

$$\Rightarrow -80 \ln\left(\frac{1}{8}\right) = t$$

$$\Rightarrow t = 143.34$$

143.34 min.

9.) (10 pt) Find $\frac{d}{dx}\int_{2x^7}^{7} \frac{\sqrt{u}}{\arctan(u)+3}\,du$. (You do not need to simplify).

$$A(x) = \int_7^x \frac{\sqrt{u}}{\arctan(u)+3}\,du$$

$$A'(x) = \frac{\sqrt{x}}{\tan^{-1}(x)+3}$$

we want $\frac{d}{dx} - A(2x^7) =$

$$\underbrace{-}_{1\text{pt}}\underbrace{\frac{\sqrt{2x^7}}{\arctan(2x^7)+3}}_{5\text{pts}} \cdot \underbrace{14x^6}_{4\text{pts.}}$$

10.) (15 pts) Suppose there are 7,000 students at Highline. Three of these students catch the highly contagious rabbit-flu and it begins to spread among the student population with a total of five students having it on the next class day. Initially, the number of cases grows at a rate proportional to the number of students that have the flu. But, officials anticipate that the number of cases diagnosed would decrease should more than 10% of the students catch the rabbit flu.

a.) (5 pts) Set-up a differential equation to model the number of rabbit-flu cases after t days at Highline.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

must find the initial growth rate

$$\underline{\frac{dP}{dt} = kP\left(1 - \frac{P}{700}\right)}$$

b.) (10 pts) Solve the differential equation from (a.). If you can't answer (a.), I will give you the solution, but it will cost you 3 points.

$$P = \frac{700}{1+Ae^{-kt}}$$

$$P(0) = 4 = \frac{700}{1+A}$$

$$\Rightarrow 4 + 4A = 700$$

$$\Rightarrow A = \frac{696}{4} = 174$$

$$P(1) = 5 = \frac{700}{1+174e^{-k}}$$

$$1 + 174e^{-k} = \frac{700}{5}$$

$$\Rightarrow 174e^{-k} = 139$$

$$\Rightarrow e^{-k} = \frac{139}{174}$$

$$\Rightarrow -k = \ln\left(\frac{139}{174}\right)$$

$$\Rightarrow k = -\ln\left(\frac{139}{174}\right)$$

$$\Rightarrow k = 0.225$$

$$\underline{P(t) = \frac{700}{1+174e^{-.225t}}}$$

c.) (2 pts) How many days until 500 students have the rabbit flu?

$$100 = \frac{700}{1+174e^{-.225t}}$$

$$1 + 174e^{-.225t} = 7$$

$$e^{-.225t} = \frac{6}{174}$$

$$\Rightarrow -.225t = \ln\left(\frac{6}{174}\right)$$

$$t = -\frac{1}{.225}\ln\left(\frac{6}{174}\right)$$

<u>15 days.</u>