**Test 1**
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Math 125 – Winter 2009

**No work = no credit**

**Name:** key

*The relationship of these* [geometric] *presumptions is left in the dark; one sees neither whether and in how far their connection is necessary, nor a priori whether it is possible.*

Georg Friedrich Bernhard Riemann (1826 - 1866)
German mathematician

Warm-ups (1 pt each) $2+3=$ 5 $-2^4=$ $-16$ $\int_{-1}^{1} 3\,dx=$ 6

1.) (1 pt) According to the difficult quote above, what did Riemann believe about the relationship between the presuppositions (assumptions) of geometry? Answer using complete sentences.

The relationship is unknown (in the dark)

2.) (10 pt) Evaluate the definite integral $\int_{-1}^{1} \frac{x^2}{\left(x^3+5\right)^2}\,dx$, if it exists.

$I = \int_{-1}^{1} \frac{x^2}{(x^3+5)^2}\,dx$ Let $u = x^3+5$

$\frac{du}{3} = \frac{3x^2 dx}{3}$

$= \int_4^6 \frac{1}{3} \cdot u^{-2}\,du$ 2 pts

$u(1) = 6$
$u(-1) = 4$ } 3 pts

$= -\frac{1}{3}\left[u^{-1}\right]_4^6$

$= -\frac{1}{3}\left(\frac{1}{6} - \frac{1}{4}\right)$

$= \frac{1}{36}$

Solution: ______________________

3.) (10 pt) Use Part 1 of the Fundamental Theorem of Calculus to find $\frac{d}{dx}\int_{4x-3}^{1}\frac{\arctan(u)}{u}\,du$.

(You do not need to simplify).

$\frac{5}{10}$ for $\arctan(1) - \frac{\arctan(4x-3)}{4x-3}$.

3 pts.

Solution: $-\frac{\arctan(4x-3)}{4x-3}\cdot 4$

4.) (4 pt) Express the integral $\int_{-2}^{3}\left(x^2-7\right)dx$ as a limit of Riemann sums (right end points). **Do not evaluate the integral**.

$\Delta x = \frac{5}{n}$

$x_i = -2 + \frac{5i}{n}$

$\int_{-2}^{3}(x^2-7)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{5i}{n}-2\right)^2-7\right]\frac{5}{n}$

Solution: ______________________

5.) (10 pts) Consider the integral $\int_0^3 \left(1+4x^3\right)\,dx$

a.) (2 pts) Evaluate the definite integral using the Fundamental Theorem of Calculus, part 2.

$$\int_0^3 (1+4x^3)\,dx = x + x^4 \Big|_0^3 \quad 3 + 81 = 84$$

b.) (4 pts) Use the <u>definition of the definite integral</u> to evaluate the definite integral given that the sum $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Furthermore, the integral is already given to you as the limit of the Riemann sums.

$$\int_0^3 \left(1+4x^3\right)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+4\left(\frac{3i}{n}\right)^3\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+4\cdot\frac{27i^3}{n^3}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3}{n}+\frac{324i^3}{n^4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3}{n}\cdot n + \frac{324}{n^4}\cdot\left(\frac{n(n+1)}{2}\right)^2\right)$$

$$= 3 + \frac{324}{4}$$

$$= 84$$

Solution: ______________________

6.) (10 pt) Consider the indefinite integral $\int \cos(x) \cdot e^{\sin(x)}\,dx$.

a.) Evaluate the indefinite integral

Let $u = \sin x$
$du = \cos x\,dx$

$$\int \cos x\, e^{\sin x}\,dx = \int e^u\,du$$
$$= e^u + C$$
$$= e^{\sin x} + C.$$

$-1$ if $-e^{\sin x} + C$

Solution: ______________________________

b.) Find the exact average value of $f(x) = \cos(x) \cdot e^{\sin(x)}$ over the interval $\frac{\pi}{6} \le x \le \frac{\pi}{2}$.

$$f_{ave} = \frac{1}{\frac{\pi}{2} - \frac{\pi}{6}} \int_{\pi/6}^{\pi/2} \cos x\, e^{\sin x}\,dx$$
$$= \frac{3}{\pi}\left[e^{\sin x}\right]_{\pi/6}^{\pi/2}$$
$$= \frac{3}{\pi}\left(e^1 - e^{1/2}\right)$$
$$= \frac{3(e - \sqrt{e})}{\pi}$$

4 pts.

$-2$ if $\frac{\pi}{3}$.

$\approx 1.0213$ ($-1$, not exact)

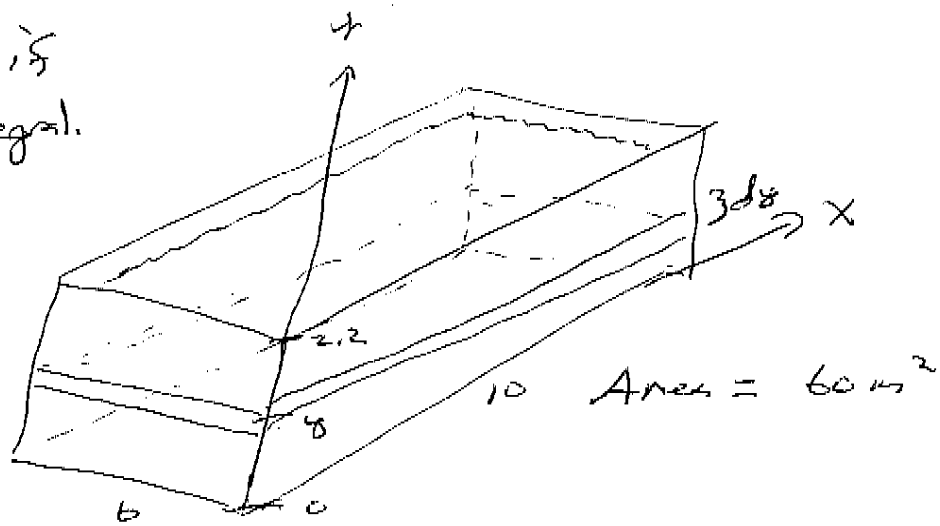
Solution: ______________________________

7.) (10 pts) A rectangular swimming pool has a length of 10m and a width of 6m. The sides are 2.2m high, and the depth is a uniform 2m (the water level is below the edge of the pool).

**Set up the integral** to find the work that is required to pump all of the water out over the side. Remember, you are not to evaluate the integral unless you want to verify that the work done is

The density of water is $1000\frac{kg}{m^3}$ and the acceleration of gravity is approximately $9.8\frac{m}{s^2}$.

6 pts is an integral.




$u = 2.2 - y$

$du = -dy$

$$\Delta W = F \cdot d$$
$$= m \cdot a \cdot d$$
$$= \rho \cdot v \cdot a \cdot d$$
$$= \rho \cdot A \cdot \Delta y \cdot a \cdot d$$
$$= 1000(60)(9.8)(2.2-y)\Delta y$$

2pts.

$$W = -\int_{2.2}^{.2} 1000(60)(9.8)\, u\, du$$
$$= \int_{.2}^{2.2} \underbrace{1000(60)(9.8)}_{588{,}000}\, u\, du$$

2 pts

2 pts.

Solution: $W = \int_0^2 1000(60)(9.8)(2.2-y)\,dy$

$1.4112 \times 10^6$ J

8.) (10 pt) Complete one of the follow two problems (That is, either (a.) or (b.)).

a.) State the Fundamental Theorem of Calculus, Part 1. Make sure to include all of the conditions and the full conclusion.

If $f$ is cont on $[a,b]$, then

$g(x) = \int_a^x f(t)\,dt$ on $[a,b]$ is

(i.) cont on $[a,b]$

(ii.) diff on $(a,b)$

(iii.) $g' = f$.

b.) Complete the following proof of Part 2 of the Fundamental Theorem of Calculus. To do this, carefully read the given portion of the proof and begin your proof where the given part ends.

Claim: If $f$ is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F$ is any antiderivative of $f$, that is, a function such that $F' = f$.

□ *Proof.*

Let $g(x) = \int_a^x f(t)\,dt$. We know from Part 1 that $g'(x) = f(x)$; that is, $g$ is an antiderivative of $f$. If $F$ is any other antiderivative of $f$ on $[a, b]$, then we know

$$\text{(i.) } F(x) = g(x) + C \text{ for } a < x < b.$$

But both $F$ and $g$ are continuous on $[a, b]$ and so, by taking limits of both sides of (i.) as $x \to a^+$ and $x \to b^-$, we see that:

$$\text{(ii.) } F(x) = g(x) + C \text{ for } a \le x \le b.$$

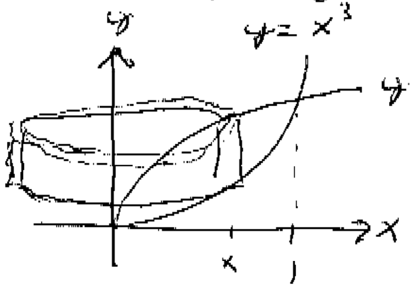
So, using equation (ii.) with $x = b$ and $x = a$, we have . . . (begin your proof here)

$$F(b) - F(a) = (g(b) + c) - (g(a) + c)$$
$$= g(b) - g(a)$$
$$= g(b)$$
$$= \int_a^b f(x)\,dx.$$

9.) (20 pts) Work two of the following three volume questions. Use the methods of disks/washers once and the method of cylindrical shells once. You may do the third for 5 extra credit points.

**Set up integrals** to find the volumes. **Do not evaluate**.

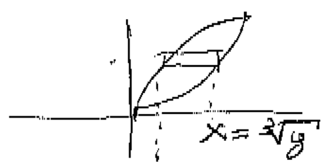
a.) The region bounded between $y = x^3$ and $y = \sqrt{x}$ is rotated about the $y$-axis.




$$\int_0^1 2\pi x(\sqrt{x} - x^3)\,dx$$

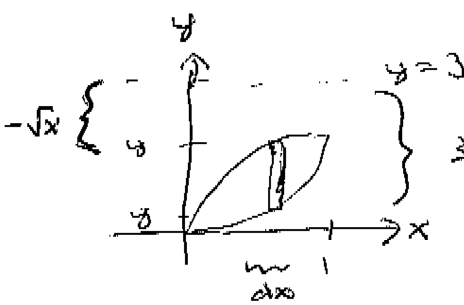
$\frac{4}{10}$ - area w/ dx

$\frac{6}{10}$ - rotated about $y$-axis




$x = y^2$ Solution: $V = \int_0^1 \pi\left((\sqrt[3]{y})^2 - (y^2)^2\right)dy$

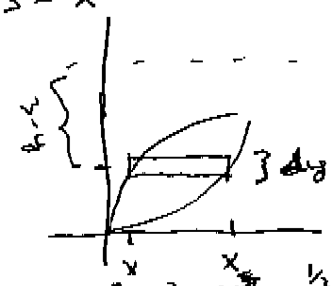
b.) The region bounded between $y = x^3$ and $y = \sqrt{x}$ is rotated about the line $y = 3$.




$$\int_0^1 \pi\left[(3 - x^3)^2 - (3 - \sqrt{x})^2\right]dx$$

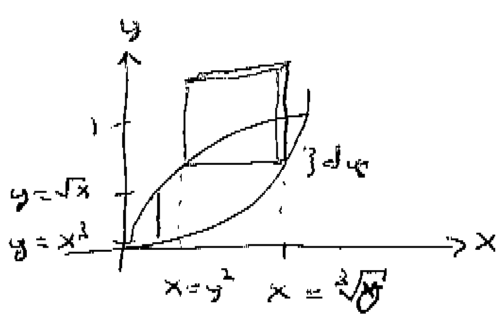
$$\int_0^1 2\pi(3 - y)(y^{1/3} - y^2)\,dy$$

-1




Solution: ______________________

c.) The base of $S$ is the region bounded between $y = x^3$ and $y = \sqrt{x}$. Cross-sections perpendicular to the $y$-axis are squares.




$$\int_0^1 (\sqrt[3]{y} - y^2)^2\,dy$$

Solution: ______________________

$\frac{8}{10}$ for $\int_0^1 (\sqrt{x} - x^3)^2\,dx$