Test 1
Dusty Wilson
Math 125 – Spring 2009

Name: KEY

If only I had the theorems!
Then I should find the proofs easily enough.

Georg Friedrich Bernhard Riemann (1826 - 1866)
German mathematician

No work = no credit

Warm-ups (1 pt each) $1+1=$ 2 $-5^2=$ -25 $\int_0^{\pi}\cos(x)\,dx=$ 0

1.) (1 pt) According to the quote (see above), what did Riemann find more easily – theorems or proofs? Answer using complete sentences.

He found the ~~th~~ Proofs more easily than thms.

2.) (10 pt) Evaluate the definite integral $\int_{-2}^{2}\frac{6x}{(x^2+7)^5}\,dx$, if it exists. ← Do I want this to be zero by symmetry

Let $u = x^2+7$
$du = 2x\,dx$
$u(-2) = 11$
$u(2) = 11$

$$= \int_{11}^{11}\frac{3\,du}{u^5} = 0$$

(ODD FCT over an interval symmetric about the y-axis).

Solution: 0

3.) (10 pt) Find $\frac{d}{dx}\int_{3x^5}^{12} \frac{\ln(u)}{1+\sin(u)}\,du$. (You do not need to simplify).

Let $A(x) = \int_{12}^{x} \frac{\ln u}{1+\sin u}\,du$

$\Rightarrow A'(x) = \frac{\ln x}{1+\sin x}$

we want $\frac{d}{dx} -A(3x^5) = -A'(3x^5)\cdot 15x^4$

Solution: $-\frac{\ln(3x^5)}{1+\sin(3x^5)} \cdot 15x^4$

4.) (5 pt) Express the integral $\int_{-1}^{7} (3x^4 - 7)\,dx$ as a limit of Riemann sums (right end points). **Do not evaluate the integral**.

$\Delta x = \frac{8}{n}$; $x_i = -1 + \frac{8i}{n}$; $f(x_i) = 3\left(\frac{8i}{n}+1\right)^4 - 7$

Solution: $\int_{-1}^{7}(3x^4 - 7)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(\frac{8i}{n}+1\right)^4 - 7\right]\cdot\frac{8}{n}$

5.) (5 pt) Consider the integral $\int_0^5 (7-2x^3)\,dx$

a.) (1 pt) Evaluate the definite integral using the Fundamental Theorem of Calculus

$$\left[7x - \frac{x^4}{2}\right]_0^5 \quad 35 - \frac{625}{2} = -\frac{555}{2}$$

b.) (4 pts) Use the <u>definition of the definite integral</u> to evaluate the definite integral given that the sum $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. Furthermore, the integral is already given to you as the limit of the Riemann sums.

$$\int_0^5 (7-2x^3)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(7-2\left(\frac{5i}{n}\right)^3\right)\frac{5}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{35}{n} - \frac{1250 i^3}{n^4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{35}{n}\sum_{i=1}^{n} 1 - \frac{1250}{n^4}\sum_{i=1}^{n} i^3\right)$$

$$= \lim_{n\to\infty}\left(\frac{35}{n}\cdot n - \frac{1250}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2\right)$$

$$= 35 - \frac{1250}{4}$$

$$= \frac{-555}{2}$$

Solution: ______________________

6.) (10 pt) A cup of coffee has temperature $95°C$ and takes 30 minutes to cool to $61°C$ in a room with temperature $20°C$. Using Newton's Law of Cooling, we can show that the temperature of coffee after t minutes is $T(t)=20+75e^{-0.02t}$. What is the average temperature of the coffee during the first half hour? (give your answer to 1 decimal place),

$$T_{ave} = \frac{1}{30}\int_0^{30}\left(20+75e^{-0.02t}\right)dt$$

$$= \frac{1}{30}\left[20t - \frac{75}{0.02}e^{-0.02t}\right]_0^{30}$$

$$= \frac{1}{30}\left[\left(600 - \frac{75}{0.02}e^{-.6}\right) - \left(0 - \frac{75}{0.02}\right)\right]$$

Solution: $T_{ave} = 76.4\,°C$

7.) (10 pt) Evaluate the indefinite integral $\int \frac{x}{\sqrt[3]{x+7}}dx$.

Let $u = x+7$
$du = dx$

$$\int \frac{x}{\sqrt[3]{x+7}}dx = \int \frac{u-7}{\sqrt[3]{u}}du$$

$$= \int\left(u^{2/3} - 7u^{-1/3}\right)du$$

$$= \frac{3}{5}u^{5/3} - 7\cdot\frac{3}{2}u^{2/3} + C$$

Solution: $\frac{3}{5}(x+7)^{5/3} - \frac{21}{2}(x+7)^{2/3} + C$

8.) (10 pts) The Gini coefficient is a measure of the distribution of wealth in a given population. A Gini coefficient of 0 would represent perfect equality (utopian socialism) and a coefficient of 1 would represent a situation where one ~~person~~ individual had 100% of the wealth. In 1950, the Gini coefficient for the USA was 0.366. Find and interpret the Gini coefficient in 1970 for the USA.

Space

To calculate the Gini coefficient (G), use the formula $G = 2\int_0^1 \left[x - L(x)\right]dx$ where $L(x)$ is the Lorenz curve which describes the cumulative proportion of total income as a function of the cumulative proportion of families below the income level.

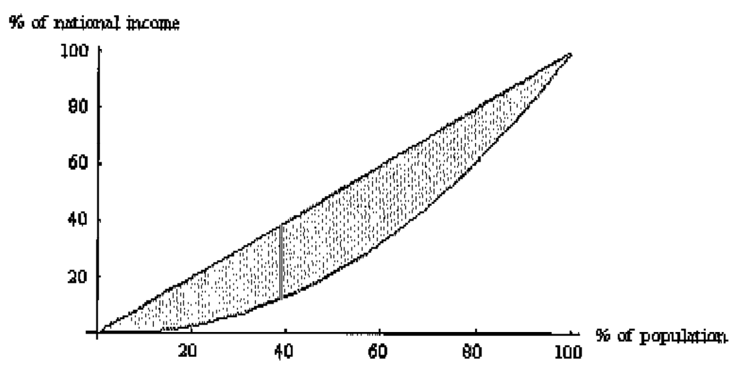
a.) Find the area between the line $y = x$ and the Lorenz curve for the USA in 1970 $L(x) = x^{2.2024}$ on the interval $0 \le x \le 1$ (the shaded area in the graph).

$$\int_0^1 (x - x^{2.2024})\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^{3.2024}}{3.2024}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3.2024}$$

$$\approx 0.1877$$



b.) The Gini coefficient is twice the shaded area found in part (a.). What is the Gini coefficient to three decimal places.

$G = 0.375$

c.) Based on your results, was income more or less equally distributed in 1970 than in 1950? Explain your answer.

Income was less equally distributed because the Gini coefficient is greater.

10.) (10 pt) Complete one of the follow two problems (That is, either (a.) or (b.)).

a.) State the Fundamental Theorem of Calculus. Make sure to include all of the conditions and the full conclusion.

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

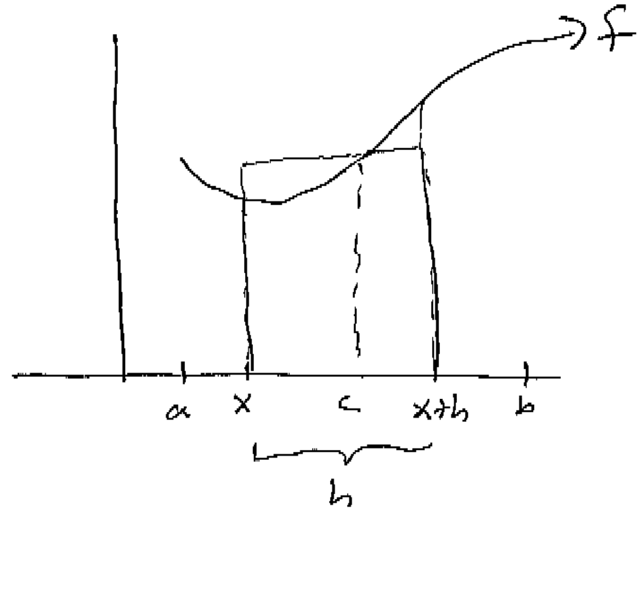
b.) Suppose f is given and satisfies the necessary conditions. Then, define the cumulative area function to be $A(x) = \int_a^x f(t)\,dt$. Show that $A'(x) = f(x)$. You may use symbols and/or picture arguments.

$$A(x+h) - A(x) = f(c)\cdot h$$ for some $c \in [x, x+h]$ by the MVT for definite integrals. (see picture)

$$\Rightarrow \frac{A(x+h) - A(x)}{h} = f(c)$$

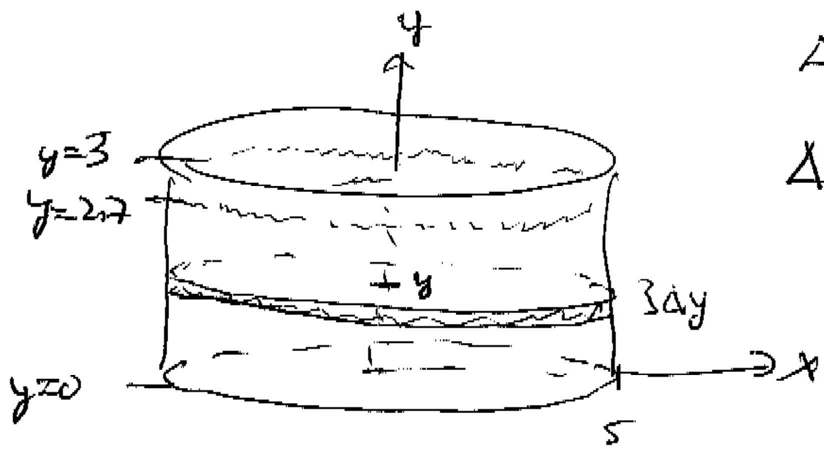
$$\Rightarrow \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} f(c)$$

$$\Rightarrow A'(x) = f(x)$$



9.) (10 pts) A circular swimming pool has a diameter of 10m, the sides are 3m high, and the depth of the water is 2.7m. **Set up the integral** to find the work that is required to pump all of the water out over the side. **Do not evaluate**.

The density of water is $1000 \, {}^{kg}/_{m^3}$ and the acceleration of gravity is approximately $10 \, {}^{m}/_{s^2}$.



$\Delta V = 25\pi \Delta y$

$\Delta W = 25(1000)(9.8)\pi(3-y)\Delta y$

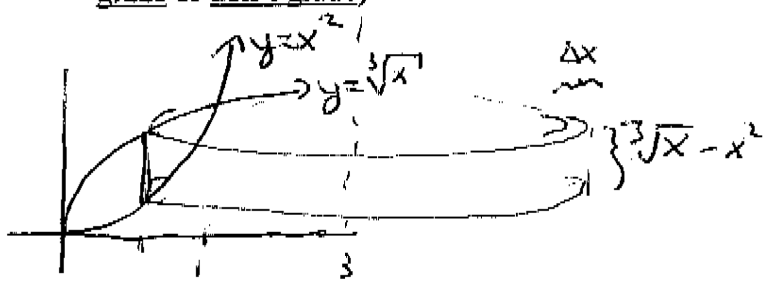
$$W = \int_0^{2.7} 25(1000)(9.8)\pi(3-y)\,dy.$$

Solution: ______________________________

11.) (20 pts) Work two of the following three volume questions. Use the methods of disks/washers and cylindrical shells. You may do the third for 5 extra credit points provided you ace the other two.

Set up integrals to find the volumes of the solids described below. **Do not evaluate**.

a.) The region bounded between $y = x^2$ and $y = \sqrt[3]{x}$ is rotated about the line $x = 3$. (Circle one: grade or don't grade)

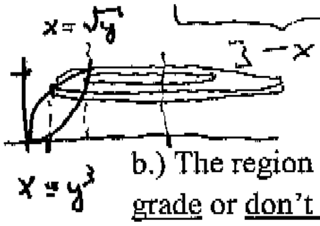


shells

$$V = \int_0^1 2\pi(3-x)(\sqrt[3]{x}-x^2)\,dx$$

slices

$$V = \pi\int_0^1 \left[(3-y^3)^2 - (3-\sqrt{y})^2\right]dy$$

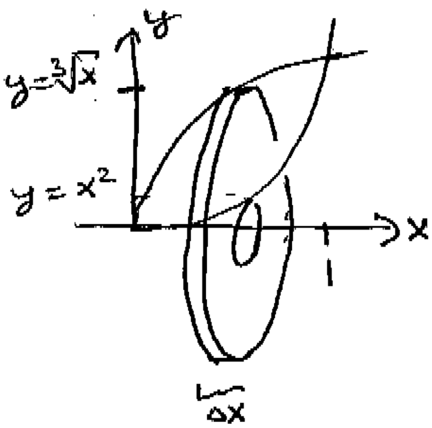


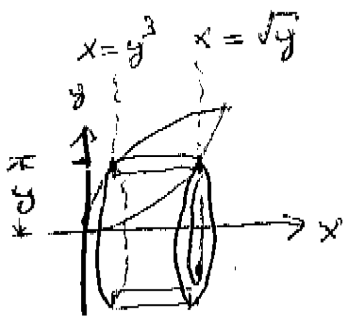
Solution (Circle one: slices or shells): ______________ $\frac{15\pi}{7}$

b.) The region bounded between $y = x^2$ and $y = \sqrt[3]{x}$ is rotated about the x-axis. (Circle one: grade or don't grade)

slices

$$V = \int_0^1 \pi\left[(\sqrt[3]{x})^2 - (x^2)^2\right]dx$$



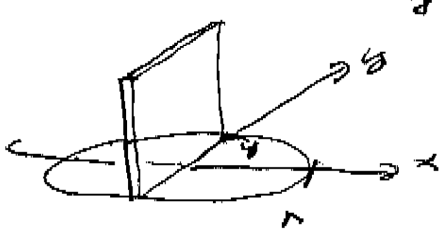


shells

$$V = \int_0^1 2\pi y(\sqrt{y}-y^3)\,dy$$

Solution (Circle one: slices or shells): ______________ $\frac{2\pi}{5}$

c.) The base of the solid S is a circular disk with radius r. Parallel cross-sections perpendicular to the base are squares. $y = \pm\sqrt{r^2-x^2}$



slices

$$V = \int_{-r}^{r} 4(r^2-x^2)\,dx$$

$\Delta V = (2y)^2\Delta x$

$= 4(r^2-x^2)\Delta x$

Solution: ______________ $\frac{16r^3}{3}$