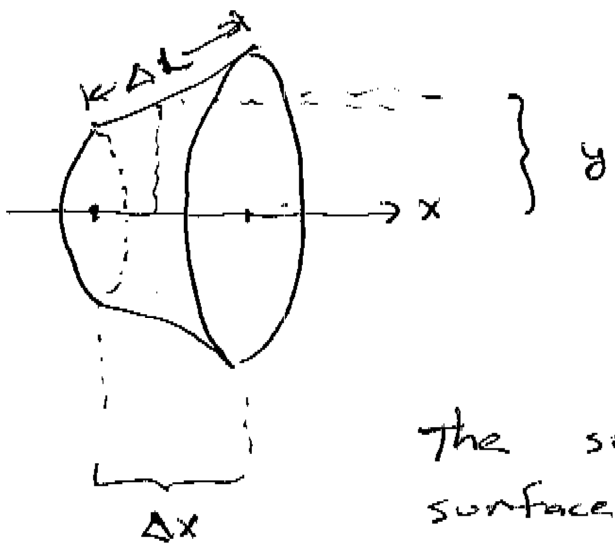
# 8.2: Area of a surface of revolution.

recall $\Delta L = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$ and

$$dL = \sqrt{1 + (f'(x))^2}\, dx$$

if we rotate about the x-axis




$$\Delta S = 2\pi y \, \Delta L$$

$$dS = 2\pi y \sqrt{1 + (y')^2}\, dx$$

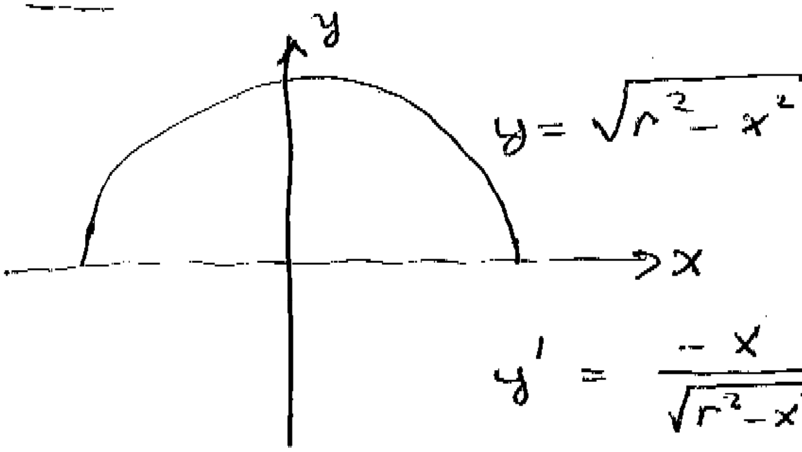
The surface area of the surface obtained by rotating $y = f(x)$ on $[a, b]$ about the x-axis...

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$$

& if $x = g(y)$ on $[c, d]$ is rotated about the y-axis

$$S = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy.$$

ex1: Verify the SA of a sphere.




$$y' = \frac{-x}{\sqrt{r^2-x^2}}$$

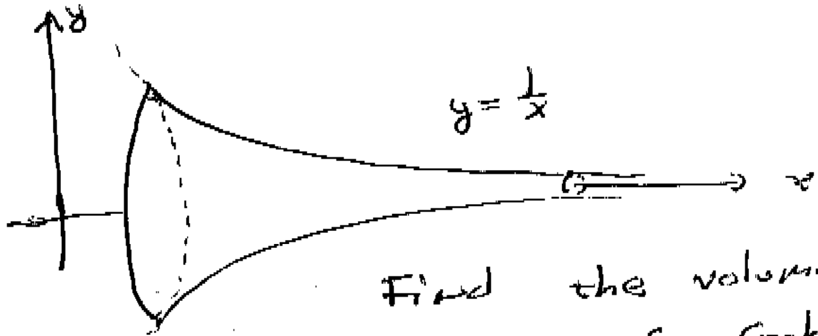
$$SA = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\left(1+\left(\frac{x}{\sqrt{r^2-x^2}}\right)^2\right)^{1/2} dx$$

$$= 4\pi\int_0^r \sqrt{r^2-x^2}\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\, dx$$

$$= 4\pi\int_0^r \cancel{\sqrt{r^2-x^2}}\cdot\frac{r}{\cancel{\sqrt{r^2-x^2}}}\, dx$$

$$= 4\pi r^2.$$

## ex2: Gabriel's Trumpet




Find the volume & surface area of Gabriel's Trumpet,

$$V = \int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \int_1^{\infty} \pi x^{-2}\, dx$$

$$= \lim_{t\to\infty}\left(\left[-\pi \frac{1}{x}\right]_1^t\right)$$

$$= \lim_{t\to\infty}\left(\pi - \frac{\pi}{t}\right)$$

$$= \pi$$

$$SA = \int_1^{\infty} 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\, dx$$

$$\frac{x^4+1}{x^4}$$

$$x^2 = \tan\theta$$

$$= \int_1^{\infty} 2\pi \cdot \frac{1}{x^3}\sqrt{x^4+1}\, dx \qquad 2x\,dx = \sec^2\theta\, d\theta$$

$$= \int 2\pi \cot^2\theta \cdot \sec\theta \cdot \sec^2\theta\, d\theta$$

# Surface Area of Gabriel's Trumpet.

$$y = \frac{1}{x} \quad ; \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

$$SA = \int_1^\infty \frac{2\pi}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$$

$$= \int_1^\infty \frac{2\pi}{x^3}\sqrt{x^4+1}\, dx$$

consider

$$* \quad I = \int \frac{2\pi}{x^3}\sqrt{x^4+1}\, dx = \int \frac{2\pi x}{x^4}\sqrt{x^4+1}\, dx$$

$\boxed{\text{Trig sub}}$ Let $x^2 = \tan\theta$

$$2x\,dx = \sec^2\theta\, d\theta$$

$$I = \pi \int \frac{\sqrt{\tan^2\theta + 1}}{\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \pi \int \frac{\sec^3\theta}{\tan^2\theta}\, d\theta$$

$$= \pi \int \frac{d\theta}{\sin^2\theta \cos\theta}$$

$$= \pi \int \frac{\cos\theta\, d\theta}{\sin^2(1-\sin^2\theta)}$$

Let $u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \pi \int \frac{du}{u^2(1-u^2)}$$

$\boxed{\text{partial fractions}}$ $$\frac{1}{u^2(1-u^2)} = \frac{A}{1+u} + \frac{B}{1-u} + \frac{C}{u} + \frac{D}{u^2}$$

$$\Rightarrow \quad 1 = A(1-u)u^2 + B(1+u)u^2 + Cu(1-u^2) + D(1-u^2)$$

$u=0: \quad D=1$

$u=1: \quad 2B=1 \Rightarrow B = 1/2$

$u=-1: \quad 2A=1 \Rightarrow A = 1/2$

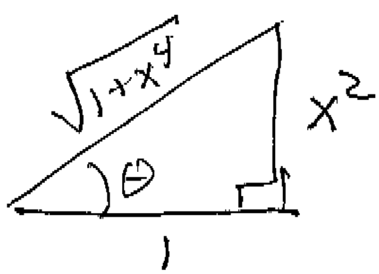
match $u^1$ coefficients: $C = 0$.

$$I = \pi \int \left( \frac{1/2}{1+u} + \frac{1/2}{1-u} + \frac{1}{u^2} \right) du$$

$$= \frac{\pi}{2} \left( \ln(1+u) - \ln(1-u) - \frac{2}{u} \right) + C$$

$$= \frac{\pi}{2} \left( \ln \left| \frac{1+u}{1-u} \right| - \frac{2}{u} \right) + C$$

$\boxed{\text{sub back}}$ $$= \frac{\pi}{2} \left( \ln \left| \frac{1+\sin\theta}{1-\sin\theta} \right| - \frac{2}{\sin\theta} \right) + C$$




$$\sin\theta = \frac{x^2}{\sqrt{1+x^4}}$$

$$I = \frac{\pi}{2}\left( \ln \left| \frac{1 + \frac{x^2}{\sqrt{1+x^4}}}{1 - \frac{x^2}{\sqrt{1+x^4}}} \right| - \frac{2\sqrt{1+x^4}}{x^2} \right) + C$$

$$= \frac{\pi}{2}\left( \ln \left| \frac{\sqrt{1+x^4} + x^2}{\sqrt{1+x^4} - x^2} \right| - \frac{2\sqrt{1+x^4}}{x^2} \right) + C$$

$*$ so $\displaystyle\int_1^\infty \frac{2\pi}{x^3}\sqrt{x^4+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{2\pi}{x^3}\sqrt{x^4+1}\,dx$

$$= \frac{\pi}{2}\lim_{t\to\infty}\left( \ln \left| \frac{\sqrt{1+t^4} + t^2}{\sqrt{1+t^4} - t^2} \right| - \frac{2\sqrt{1+t^4}}{t^2} \right) - \#$$

$$= \infty.$$

So, the SA on Gabriel's Trumpet is infinite.

... now, the easy way

$$SA = \int_1^\infty \frac{2\pi}{x^3}\sqrt{1+x^4}\,dx \geq 2\pi \int_1^\infty \frac{\sqrt{x^4}}{x^3}\,dx$$

$$= 2\pi \int_1^\infty \frac{dx}{x} \quad \text{diverges by p-test.}$$

So, the SA is $\infty$ (diverges) by comparison

SA of the surface formed by rotating
$y = \sqrt[3]{x}$ on $1 \le y \le 2$.

method 1: $\displaystyle\int_1^8 2\pi x \sqrt{1 + \left(\frac{1}{3x^{2/3}}\right)^2}\, dx$

method 2: $\displaystyle\int_1^2 2\pi y^3 \sqrt{1 + (3y^2)^2}\, dy$